ARTICLE IN PRESS
FSS:7220

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An optimization problem on the image set of a (max, min) fuzzy operator ☆

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Received 12 January 2016; received in revised form 16 January 2017; accepted 2 May 2017

Abstract

The paper deals with the unsolvability of (max, min) linear equation systems with coefficients in the unit interval (for fuzzy systems), or in an arbitrary real interval (general case). If the system has no solution, then the nearest vector to the right-hand side of the system for which the system is solvable is computed. A polynomial algorithm for the problem is presented. The method is illustrated by numerical examples.

Keywords: (max, min) fuzzy algebra; (max, min) equations; (max, min) inequalities; (max, min) linear systems

1. Introduction

Algebraic structures in which a pair of binary operations $(\oplus, \otimes)$ plays the same role as addition and multiplication in the classic linear algebra have been found in the literature since the 1960s. These operations can be extended to Cartesian products of a finite number of such sets, which enables formulating various "$(\oplus, \otimes)$-linear" problems, in which "$(\oplus, \otimes)$-linear" functions occur. The $\oplus$-operation is usually a commutative semi-group operation and the $\otimes$-operation is either a commutative group or semi-group operation. Moreover, the distributive law with respect to $\oplus, \otimes$ is assumed. Such structures with $(\oplus, \otimes) = (\max, +)$ appeared probably for the first time, e.g., in [4,17] and were later further developed, generalized and applied by other authors, e.g., in [1,5,11,14,16]. The relatively recent state of the art in the theory for the case $(\oplus, \otimes) = (\max, +)$ can be found in [2]. Since the operations max and min are used to express the membership function values of the union and intersection of fuzzy sets, (max, min)-linear problems have found applications also in fuzzy set theory, see, e.g., [3,7,8,13,16]. Fuzzy relational systems have been used in some optimization problems, see [6,9,10,12,15,18,19].

☆ The support of the Grant Agency of Excellence UHK FIM is gratefully acknowledged.
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http://dx.doi.org/10.1016/j.fss.2017.05.004

The problem we are going to solve in this paper concerns the solvability of systems of one-sided fuzzy relational equations. If the system has no solution, then the nearest right-hand side for which the system is solvable, is found. The problem of approximate solutions of unsolvable relational equation systems by minimizing the Euclidean distance between the right and left hand sides of the equations was considered in [15], where a Newton-type iterative procedure was suggested.

In our paper, we propose an alternative different approach, which uses the $||...||_\infty$ norm. The proposed method is a derivative-free direct polynomial method, which takes advantage of the (max/min)-separability of the objective function and can be extended to problems with upper and lower bounds on variables. Besides, the proposed method can be adjusted to problems with integer variables.

We start with a (max, min) linear equation system of the form

$$\max_{1\le j\le n}\big(\min(a_{ij}, x_j)\big) = \hat{b}_i,\ i = 1, \ldots, m, \tag{1}$$

where $a_{ij}, \hat{b}_i$ belong to a given interval R of real numbers. The unknown values of the variables x_j are supposed to be in R, as well. In fuzzy applications, R is the unit real interval [0, 1]. However, the problem is formulated (and solved) here for an arbitrary real interval R, possibly even for $(-\infty, +\infty)$.

Remark 1.1. The proposed method can be adjusted to cases, in which variables $x_j,\ j \in J$ have to satisfy some further constraints. For instance, we can require that x_j belong to a closed interval (e.g. $x_j \in [0, 1]$), or consider only integer x_j, or x_j belonging to a finite set.

If (1) has no solution, we will find the right-hand side that is nearest to $\hat{b}$ for which the system is solvable. The exact formulation is given in the next section.

From now on, given fixed natural numbers m, n we use the notation $I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$.

The motivation for (1) is illustrated by the following three examples: from fuzzy set theory, computer network security, and operations research.

Example 1.1. Assume that $G_1, \ldots, G_m$ are given fuzzy goals with membership functions $\mu_i,\ i \in I$ having a common finite support J. Write $\mu_i(j) = a_{ij} \in [0, 1]$, for every $i \in I,\ j \in J$. Let X be a fuzzy variable with unknown membership function $\mu : J \longmapsto [0, 1]$. The values of the membership functions of fuzzy intersections $X \cap G_i,\ i \in I$ are defined as follows

$$\mu_{G_i \cap X}(j) = \min\big(\mu_{G_i}(j), \mu(j)\big) = \min(a_{ij}, x_j),\ \text{for every } i \in I,\ j \in J.$$

The task is to find X (that is, to find the values $\mu(j) = x_j,\ j \in J$) in such a way such that the maximum membership value of the intersection membership function $\min\big(\mu_{G_i}(j), \mu(j)\big) = \min(a_{ij}, x_j)$, would be equal to a prescribed value $\hat{b}_i$ for every $i \in I$. In other words, we are looking for $x_j \in [0, 1],\ j \in J$ satisfying the equation system

$$\max_{j\in J}\big(\min(a_{ij}, x_j)\big) = \hat{b}_i,\ i \in I.$$

If the equation system has no solution, we wish to find that right-hand side which is the nearest to $\hat{b}$ for which the equation system is solvable.

Example 1.2. In a computer network, we consider servers $S_1, \ldots, S_m$ and data storages $D_1, \ldots, D_n$. For every $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $a_{ij} \in [0, 1]$ denotes the security level of the communication connection of S_i with D_j. The task is to complete the computer network by logical connections of the data stores with the logical unit L with security level x_j for every D_j. For security reasons, x_j must be chosen in such a way that every server S_i is connected with L via data storage units at the prescribed maximal security level $\hat{b}_i$. That is, $\max_{j\in J} \min(a_{ij}, x_j) = \hat{b}_i$ should hold for every $i \in I$. If these requirements cannot be satisfied, then the prescribed values $\hat{b}_i$ must be modified to provide the solvability of the equation system. To save costs, the modification of $\hat{b} = \big(\hat{b}_1, \hat{b}_2, \ldots \hat{b}_m\big)^T$ should be as small as possible.

In mathematical notation, we have to solve the optimization problem based on the unsolvability of (1).