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Fuzzy actions

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Abstract

This paper generalizes (fuzzifies) actions of a monoid or group on a set to deal with situations where imprecision and uncertainty are present. Fuzzy actions can handle the granularity of a set or even create it by defining a fuzzy equivalence relation on it.

Keywords: Fuzzy action; Fuzzy mapping; T-indistinguishability operator; Fuzzy subgroup

1. Introduction

Actions of a group or monoid G on a set I are a very useful tool in many branches of Mathematics and Computer Science. Paradigmatic examples are the actions of subgroups of the general linear group $\mathrm{GL}(n, \mathbb{R})$ on the vector space $\mathbb{R}^n$, the action of the projective linear group $\mathrm{PGL}(n+1, \mathbb{R})$ on the projective space $\mathbb{P}^n(\mathbb{R})$ and the action of the symmetry groups of regular polygons, friezes and wallpapers [10]. These examples are of geometric nature and are special cases of the concept of Geometry introduced by Klein in his Erlangen Program [6], where a Geometry is defined as a set and a group acting on it.

Actions are also useful in abstract algebra. In group theory, for example, translation and conjugation of a group on itself help prove important results such as the Lagrange Theorem [8].

In Computer Science, actions appear in Automata Theory and in Pattern Recognition, especially in character recognition problems [4] [11], where an image x can be compared with a prototype p by searching for an element g of a specific group such that the action of g on x transforms it into p ($gx = p$).

There are situations where imprecision, lack of accuracy or noise have to be taken into account or must be added to the problems to be solved. In the last example of character recognition, for instance, it is unlikely that we could find g of a "reasonable" group acting on the set of images or characters in such a way that gx is exactly p. In fact we expect to say that x corresponds to the character p when we can find g with gx only close or similar to p. In these cases, the action must be relaxed and allow it to consider imprecision and inaccuracy.

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In this paper we present and develop the concept of fuzzy action that generalizes the idea of action of a group or monoid G on a set I. The action of an element $g \in G$ on an element $x \in I$ is not a precise element of I, but a fuzzy set encapsulating the imprecision given by the granularity of the system. According to Zadeh, granularity is one of the basic concepts that underlie human cognition [18] and the elements within a granule 'have to be dealt with as a whole rather than individually' [17].

> Informally, granulation of an object A results in a collection of granules of A, with a granule being a clump of objects (or points) which are drawn together by indistinguishability, similarity, proximity or functionality [18].

In fact, it will be proved that a fuzzy action α on a set I generates a fuzzy equivalence relation (an indistinguishability operator) E_I on I in a natural way and that from a crisp action on I and an indistinguishability operator on I satisfying an invariant condition with respect to the action (see Definition 3.19) a fuzzy action derives.

There are a few number of attempts to fuzzify actions on sets previous to this paper. Haddadi [5] and Roventa and Spircu [16] study fuzzy actions of fuzzy submonoids and fuzzy subgroups from an algebraic point of view. Lizasoain and Moreno [9] generalize results of [4] and [11] for the comparison of deformed images. In these papers the approach is rather different to the one proposed here.

The paper is organized as follows: After this introductory section, a section of preliminaries contains the basic definitions and properties of fuzzy subgroups, indistinguishability operators and fuzzy mappings needed on the paper. The composition of two mappings differs from the usual one in the sense that compatibility with the intermediate indistinguishability operator is required (cf. Definition 2.10). Consequently, subsequent properties of Section 2 are new. Section 3 contains the main results of the paper. The first subsection of Section 4 generalizes fuzzy actions of a group or monoid to fuzzy actions of fuzzy groups or monoids and specializes it to the restriction of a crisp action of G to a fuzzy subgroup or submonoid of G. The second subsection of Section 4 contains a couple of examples.

2. Preliminaries

This section contains some definitions and properties related to fuzzy subgroups and T-indistinguishability operators that will be needed later. Some definitions and properties of fuzzy maps needed in the paper will be stated.

Definition 2.1. A t-norm (triangular norm) is a binary operation T on the unit interval [0, 1] such that for all $x, y, z \in [0, 1]$ tho following axioms are satisfied:

1. $T(x, y) = T(y, x)$ (Commutativity)
2. $T(x, T(x, y)) = T(T(x, y), z)$ (Associativity)
3. $T(x, y) \leq T(x, z)$ whenever $y \leq z$ (Monotonicity)
4. $T(x, 1) = x$ (Boundary condition).

A t-norm is called left-continuous if and only if it is left-continuous with respect to both variables.

Throughout the paper T will denote a given t-norm. Also the groups and sets can have either finite or infinite cardinality.

Fuzzy subgroups were introduced by Rosenfeld [15] as a natural generalization of the concept of subgroup and have been widely studied [12].

Definition 2.2. Let G be a group and μ a fuzzy subset of X. μ is a T-fuzzy subgroup of G if and only if $T(\mu(g), \mu(h)) \leq \mu(gh^{-1})$ $\forall g, h \in G$.

Proposition 2.3. *Let G be a group, e its identity element and μ a fuzzy subset of G such that $\mu(e) = 1$. Then μ is a T-fuzzy subgroup of G if and only if $\forall g, h \in G$ the following properties hold*

1. $\mu(g) = \mu(g^{-1})$
2. $T(\mu(g), \mu(h)) \leq \mu(gh)$.

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