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Fuzzy sets within finitely supported mathematics

Andrei Alexandru, Gabriel Ciobanu

Romanian Academy, Institute of Computer Science, Blvd King Carol I no. 8, 700505 Iaşi, Romania

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Abstract

The classical theory of fuzzy sets is extended to a recently developed framework named Finitely Supported Mathematics in order to characterize fuzzy sets over infinite universes in a finitary manner by involving the concept of "finite support". We prove some algebraic properties of the new fuzzy sets within Finitely Supported Mathematics (including some that cannot be obtained in Zermelo–Fraenkel mathematics), and introduce operations and extension principles over these fuzzy sets. We introduce a specific (infinite) membership-degree association, and connect it to the notions of invariant complete lattices and invariant monoids in Finitely Supported Mathematics.

Keywords: Finite support; Invariant complete lattices and monoids; Membership-degree association

1. Introduction

The theory of fuzzy sets was first presented in [9] by extending the classical theory of sets. It represents a framework for rigorously studying the concepts of uncertainty and vagueness. Informally, fuzzy sets are sets where all their elements have degrees of membership belonging to the real unit interval $[0, 1]$. Applications of this theory can be found in various areas of different sciences such as algebra, logic, real analysis, control theory, topology, operations research, artificial intelligence, robotics, decision theory, expert systems and pattern recognition. A detailed collection of applications of fuzzy sets can be found in [10].

Finitely Supported Mathematics (FSM) is presented extensively in the recent book [2]. It extends the classical Zermelo–Fraenkel mathematics (ZF), aiming to model infinite structures in terms of finitely supported objects. FSM is inspired by the axioms of Fraenkel–Mostowski set theory (FM) which are the axioms of Zermelo–Fraenkel set theory with atoms (ZFA) over an infinite set of atoms together with an axiom of finite support claiming that each element in an arbitrary set is finitely supported [6]. The construction of the Cumulative Hierarchy Fraenkel–Mostowski universe $FM(A)$ of all FM sets (i.e. the class of all sets constructed according to the FM axioms) is presented in [6] as a model of FM set theory, and it is inspired by the construction of the universe of all admissible sets over an arbitrary

E-mail addresses: aalexandru@iit.tuiasi.ro (A. Alexandru), gabriel@info.uaic.ro (G. Ciobanu).

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collection of atoms [5]. The FM sets represent a generalization of hereditary finite sets because any FM set is actually an hereditary finitely supported set.

Rather than using a non-standard set theory, someone could alternatively work with nominal sets which are well-defined in the classical ZF framework as usual sets endowed with some group actions satisfying a finite support requirement [8]. There exists also an alternative definition for nominal sets in the FM framework; they can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all finitary permutations of atoms). These two ways of defining nominal sets finally lead to similar properties [2]. We use the terminology "invariant" instead of "nominal" in order to establish a connection between the approaches in the FM framework and in the ZF framework.

FSM represents the ZF mathematics rephrased in terms of finitely supported structures, where the set of atoms should be infinite (not necessarily countable). In FSM we use either 'invariant sets' or 'finitely supported sets' instead of simply 'sets'. Informally, FSM is a theory of 'invariant algebraic structures'. FSM is not the theory of nominal sets from [8] presented in a different manner. However, the theory of nominal sets [8] could be considered as a tool for defining FSM. The connections between FSM, the theory of nominal sets, the theory of generalized nominal sets, the theory of FM sets and the permutative models of ZFA are described in [2]. Several connections between FSM, admissible sets, Gandy machines and logical notions of Tarski are mentioned in [2].

The general principle of constructing FSM is that all the structures should be finitely supported. As a consequence, we cannot obtain a property in FSM only by involving a ZF result without an appropriate proof reformulated according to the finite support requirement. Moreover, as we proved in [2], not every ZF result can be directly reformulated in FSM in terms of finitely supported objects. This is because given an invariant set, some of its subsets may be non-finitely supported. A related example is represented by a simultaneously infinite and coinfinite subset of the invariant set of all atoms.

As presented in [2], FSM deals with a more relaxed notion of infiniteness. Intuitively, in FSM we can model infinite structures by using only a finite number of characteristics. More exactly, in FSM we admit the existence of infinite structures, but for an infinite structure we ascertain that only of a finite family of its elements is "really important" in order to characterize the related structure, while the other elements are somehow "similar". This means that we associate to each object a finite family of elements characterizing it, which is called "finite support". As a simple intuitive example, in a lambda-calculus interpretation, the finite support of a lambda term is represented by the set of all "free names" of that term; the other names can be renamed without affecting the essential properties of the lambda-term.

The goal of this paper is to develop the basics of the theory of fuzzy sets within FSM. Our approach allow to characterize fuzzy sets over infinite universes in a finitary manner. Informally, we extend the concept of fuzzy sets from "a finite framework" to "an infinite, but finitely supported framework". Thus, instead of (finite) fuzzy membership functions defined on a ZF universe of discourse, we consider and study (infinite) finitely supported fuzzy membership functions defined over an FSM universe of discourse. The properties of such infinite FSM fuzzy membership functions over infinite universes of discourse are expressed in a finite manner by involving the concept of "finite support". We obtain new order properties of fuzzy sets than cannot be obtained in ZF (e.g., Proposition 35). This approach provides a first step in developing a computable framework for (infinite) fuzzy sets.

2. Invariant sets

Let A be a fixed infinite set in ZF. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$"). Thus, the theory of invariant sets makes sense both in ZF and in ZFA, and 'ZF' can be replaced by 'ZFA' in our statements.

A *transposition* is a function $(a\,b) : A \to A$ defined by $(a\,b)(a) = b$, $(a\,b)(b) = a$ and $(a\,b)(n) = n$ for $n \neq a, b$. A *permutation* of A is generated by composing finitely many transpositions. Let S_A be the set of permutations of A, namely the set of bijections on A which leave unchanged all but finitely many elements.

Definition 1. Let X be a set in ZF.

1. An S_A-*action* on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set* is a pair $(X, \cdot)$, where X is a set in ZF and $\cdot : S_A \times X \to X$ is an S_A-action on X.

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