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FSS:7327

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A graded notion of functionality

Martina Daňková

Centre of Excellence IT4Innovations, division of the University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

Received 9 June 2017; received in revised form 29 September 2017; accepted 11 November 2017

Abstract

We will introduce the notion of functionality as a graded property of a fuzzy relation. We will extend known results relating to functional fuzzy relations to their graded counterparts and provide new properties w.r.t. various kinds of relational compositions. As a contribution to the theory of fuzzy control, we will moreover provide an example of use of graded properties in establishing implicative models of fuzzy rule bases where the non-graded property fails.

Keywords: Fuzzy function; Functional fuzzy relation; Fuzzy control; Fuzzy Class Theory; Non-standard mathematics

1. Introduction

The notion of functional fuzzy relations [1,2], known also as the uniqueness property [3–5] or partial function [6,7] (where the relations are in addition extensional), is connected with early stages of fuzzy set theory and its applications in the so called “IF–THEN” rules; see e.g. [8,7,9]. During the development of fuzzy set theory, the functionality has been several times generalized, up to the point when Gottwald [3, Definition 3.1] introduced degrees of functionality. Further variations of graded functionality were later proposed by Demirci [10, property $f2$ in Definition 3.1] or by means of a cardinality of α-cuts by Stella and Guido in [11].

In this work, we follow up the approach of Gottwald [3], which is a straightforward and transparent generalization of the classical set-theoretical notion. His formalization can capture a partial fulfillment of the functionality property expressed by a degree from the scale for truth values—the so called *gradualness* of the property. Let us illustrate it on a simple example: let us take a crisp binary relation F on reals and examine its functionality. The relation F is said to be functional if and only if

$$[\![(x = x') \,\&\, F(x, y) \,\&\, F(x', y') \to (y = y')]\!] = 1, \text{ for all } x, x', y, y' \in [a, b],$$

E-mail address: martina.dankova@osu.cz.

https://doi.org/10.1016/j.fss.2017.11.010
0165-0114/

Please cite this article in press as: M. Daňková, A graded notion of functionality, Fuzzy Sets Syst. (2017), https://doi.org/10.1016/j.fss.2017.11.010

where & and → are interpreted as any t-norm and any residual implication,[1] respectively. But what if we have to deal with a defective relation F' that is identical with a functional F up to a single point c of the input space? For example let $(c, 5) \in F'$ and $(c, 5.001) \in F'$. Then F' is not functional in the above sense:

$$[\![\underbrace{(c = c) \,\&\, F'(c,5) \,\&\, F'(c,5.001)}_{=1} \to \underbrace{(5 = 5.001)}_{=0}]\!] = 0.$$

Nevertheless, it can be considered as nearly functional, which can be naturally expressed by a degree taken as the infimum over all values $[\![(x \approx x') \,\&\, F'(x,y) \,\&\, F'(x',y') \to (y \approx y')]\!]$, where we have to replace = by some fuzzy equality (proximity or some other appropriate relation of indistinguishability) ≈. Depending on the suitable choice of ≈, the expression would evaluate to a degree close to 1.

In fact, all the used notions within this paper will enjoy the gradualness, and therefore we have chosen Fuzzy Class Theory (FCT) introduced in [12] as our framework. This theory has been developed to provide all necessary tools for logic-based constructions of fuzzy mathematics. It is an extension of a well-founded fuzzy logic (encompassing the logic of left-continuous t-norms) and admits models of all fuzzy sets whose membership functions take values from the interval [0, 1]. For details and a brief explanation of the logic-based methodology in fuzzy mathematics consult [12]. We only mention three main characteristics of the logic-based approach to fuzzy mathematics:

1. We work within a specific formal fuzzy logic,[2] in particular any logic at least as strong as the fuzzy logic MTL_Δ of left-continuous t-norms (see Appendix A).
2. Fuzzy sets (relations) are treated as objects in a formal language, leaving aside a direct interpretation in terms of truth degrees.
3. Statements about the objects of the interest are in the form of *graded theorems*.

Gradualness is the main characteristic of theorems proved in this framework; accordingly we use the term *graded theorems*. As already mentioned in [13], it means that instead of the usual statement:

If φ is provable then ψ is provable — a classical theorem,

we search for a more informative and general form (not equivalent) of this statement:

$\varphi \to \psi$ is provable — a graded theorem,

where → is interpreted as a fuzzy logic implication. Recall that $[\![\varphi \to \psi]\!] = 1$ if and only if $[\![\varphi]\!] \leq [\![\psi]\!]$.

For the sake of completeness, we have to take into account the fact that fuzzy logics generally have two conjunctions (∧, &), where the lattice conjunction ∧ is idempotent, i.e., $(\varphi \wedge \varphi) \leftrightarrow \varphi$, while the strong conjunction &, in general, only satisfies $(\varphi \,\&\, \varphi) \to \varphi$. Therefore, we analyze how many times we need to apply an antecedent φ to prove the consequent ψ, which is coded in a parameter n:

$\varphi^n \to \psi$ is provable — a graded theorem, where $\varphi^n \equiv_{\mathrm{df}} \underbrace{(\varphi \,\&\, \ldots \,\&\, \varphi)}_{n\text{-times}}$.

For example, $0.9 \not\leq 0.7$, but with Łukasiewicz t-norm $0.9 \otimes 0.9 \otimes 0.9 = 0.9^3 \leq 0.7$, and such situations appear in graded theorems very often [14]. We refer to a graded theorem (7) below, where we have $n = 2$ and it is not provable for $n = 1$.

As a matter of fact, the meta-language that we are using when speaking about provable formulae over some fuzzy logic can be viewed as the formal language of the classical mathematical logic (obviously by the fact that a given formula is either provable or not). The main difference between a classical theorem and a graded theorem can be expressed as follows [15]:

"We write classically, but we think in grades."

[1] We deal only with truth values {0, 1} in this particular example and therefore the residual implication need not be derived from the same t-norm interpreting &.

[2] The notion of fuzzy logic does not represent here a wide range of applications as is usual in engineers' papers, but it denotes a formal logic (of specified order and type) having its syntax and semantic.

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