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A note on a generalized Frank functional equation

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Abstract

We study a generalized Frank functional equation in the broader framework of associative aggregation functions and show that, up to the two projections, we obtain exactly the same set of solutions as in the original paper M. J. Frank (1979) [11].

Keywords: Frank functional equation; Copula; Triangular norm; Aggregation function

1. Introduction

The Frank functional equation studied in [11] is a fundamental equation for binary copulas C [35] (see also [9,16,29]), i.e., for functions $C\colon [0,1]^2 \to [0,1]$ which are 2-increasing and satisfy $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for each $x \in [0,1]$. Let us quote verbatim from the beginning of the paper [11]:

> Consider the set of continuous two-place functions F from the unit square $[0,1]^2$ to the unit interval $[0,1]$ which satisfy the boundary conditions
>
> $$F(0,x) = F(x,0) = 0, \qquad F(1,x) = F(x,1) = x, \quad \text{for every } x.$$
>
> Given F, define the function $F^\wedge$ on $[0,1]^2$ via

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$$F^{\wedge}(x,y) = x + y - F(x,y).$$

The main purpose of this paper is to find all functions F such that *both* F and $F^{\wedge}$ are associative.

Then the author continues with a description of the motivation for his research which came from copulas.

Observe first that associative functions as considered in [11] are exactly I-semigroup operations on $[0,1]$ which were completely characterized in [27]. As a consequence, each such function $F\colon [0,1]^2 \to [0,1]$ in [11] is a (continuous) t-norm [32], i.e., F is symmetric, associative, monotone non-decreasing and satisfies $F(0,x) = F(x,0) = 0$ and $F(1,x) = F(x,1) = x$ for each $x \in [0,1]$. There is an interesting link between binary copulas and t-norms in the sense that the class of 1-Lipschitz t-norms coincides with the class of associative binary copulas [28].

Moreover, one can find several variants and generalizations of the Frank functional equation in the literature, some of them leading to the same set of solutions:

- In copula theory, the function $F^{\wedge}$ as given above is called the dual copula of F and usually denoted by F^*, and then the Frank functional equation means the simultaneous associativity of a copula F and its dual F^*.
- The same problem in the broader class of quasi-copulas [2,13] has the same set of solutions.
- In the framework of triangular norms [1,17,19,26,31,33,34] the Frank functional equation can be formulated as follows: characterize all t-norms $T\colon [0,1]^2 \to [0,1]$ and all t-conorms $S\colon [0,1]^2 \to [0,1]$ such that

$$T(x,y) + S(x,y) = x + y \qquad \text{for all } (x,y) \in [0,1]^2, \tag{1.1}$$

  and again the set of solutions is the same. We shall call (1.1) the (original) Frank functional equation.
- Also in the set of uninorms and nullnorms one obtains the same set of solutions [7].
- In [22] it was shown that, when studying the principle of inclusion and exclusion for fuzzy sets, the only solutions of a similar equality (which reduces to the Frank functional equation (1.1) in the case of two sets) are the minimum and the product. This study could be interpreted as an attempt to create a "multivariate" Frank functional equation.
- In the context of binary aggregation functions $A\colon [0,1]^2 \to [0,1]$ (see [3,15]), i.e., monotone non-decreasing functions with $A(0,0) = 0$ and $A(1,1) = 1$, there is another concept of duality (which differs from the duality of (quasi-)copulas): the dual $A^d\colon [0,1]^2 \to [0,1]$ of an aggregation function $A\colon [0,1]^2 \to [0,1]$ is given by $A^d(x,y) = 1 - A(1-x, 1-y)$. Note that the dual T^d of a t-norm T is a t-conorm, and vice versa. A modified Frank functional equation can be formulated as: find all t-norms T satisfying

$$T^d = T^*. \tag{1.2}$$

- In [1] the Frank functional equation (1.1) was generalized by replacing the standard addition of real numbers by some function $\Phi\colon [a,b]^2 \to \mathbb{R}$. This generalized equation was solved in [8] for the two special cases $\Phi_1(x,y) = \min(x,y) + k \cdot \max(x,y)$ and $\Phi_2(x,y) = k \cdot \min(x,y) + \max(x,y)$ with $k \in \,]0,1[$.
- Some other generalizations of the problem where $[0,1]$ is replaced by $[0,\infty]$ or by a finite chain can be found in [14,25].

Without going too deeply into details, recall that the general solution of the Frank functional equation is an ordinal sum $F = (\langle a_k, b_k, F_k\rangle)_{k\in K}$, where K is a (possibly empty) countable index set, $(]a_k, b_k[)_{k\in K}$ a set of pairwise disjoint open subintervals of $[0,1]$, and each $F_k\colon [0,1]^2 \to [0,1]$ a member of the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ given by

$$T^{\mathbf{F}}_\lambda(x,y) = \begin{cases} \min(x,y) & \text{if } \lambda = 0,\\ x \cdot y & \text{if } \lambda = 1,\\ W(x,y) & \text{if } \lambda = \infty,\\ \log_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{otherwise,} \end{cases}$$

in the sense that for each $k \in \mathbb{N}$ there is some $\lambda_k \in \,]0,\infty]$ such that $F_k = T^{\mathbf{F}}_{\lambda_k}$, and where $W\colon [0,1]^2 \to [0,1]$ is given by $W(x,y) = \max(x+y-1, 0)$.

Note that each solution F of the Frank functional equation and, in particular, each Frank t-norm $T^{\mathbf{F}}_\lambda$ is also a copula and, therefore, both supermodular [4,18,24] and 1-Lipschitz.