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Supermigrativity of aggregation functions

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Abstract

A functional inequality, called supermigrativity, was recently introduced for bivariate semi-copulas and applied in various problems arising in the study of aging properties of stochastic systems. Here, we revisit this notion and extend it to the case of aggregation functions in higher dimensions. In particular, we show how supermigrativity can be expressed via monotonicity of a function with respect to logarithmic majorization ordering of real vectors. Various alternative characterizations of supermigrativity are illustrated, together with some of its weaker versions. Several examples show similarities and differences between the bivariate and the general case.

Keywords: Aggregation functions; Copulas; Functional inequalities; Multicriteria decision making; Supermigrativity

1. Introduction

When the main interest is to describe and predict a system with d components, it is often convenient to represent its behavior in the language of probability theory by assuming the existence of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$, defined on a suitable probability space, such that X_i may interpret the uncertainty of the i-th system component.

In many cases, the study of a random vector $\mathbf{X}$ can be carried out by representing its probability joint distribution function $F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(X_1 \le x_1, \ldots, X_d \le x_d)$ as a composition of the marginal distributions $F_1, \ldots, F_d$ and a copula C, via the formula $F_{\mathbf{X}} = C(F_1, \ldots, F_d)$ due to the celebrated Sklar's Theorem (see, e.g., [17,24,29,31]). Moreover, it is also of interest to study the aging properties of $\mathbf{X}$, namely those properties that can help interpreting the evolution of the system at different future times (see, e.g., [28,34]).

Such studies are related to various real situations. For instance, one can interpret $\mathbf{X}$ as the vector of lifetimes of the components of an engineering disposal and, hence, the aging properties serve to indicate possible strategies in presence of the wear-out of the system. In another context, $\mathbf{X}$ can be related to lifetimes of individuals (linked, for instance, in a partnership) and the behavior of the system in time may help in the pricing of joint life insurance policies.

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While univariate notions of aging are by-now classical in the literature, when dealing with the analysis of dependent lifetimes, analogous definitions are rather controversial. A seminal contribution was provided in [5], where the non-trivial interactions between dependence properties (as described by the copula) and the aging properties were considered. Furthermore, this latter work highlighted that:

- a possible framework to study aging properties of a vector of lifetimes, in particular in the exchangeable case, is the class of semi-copulas (see, e.g., [14,16,35]), which are (fuzzy) connectives that generalize both copulas and triangular norms and have also been used in several fuzzy integrals (see, e.g., [26,27]);
- notions of multivariate aging can be expressed in terms of functional inequalities among semi-copulas (see, e.g., [4,5,11]), which have been also developed in various related problems (see, for instance, [10,20,21]).

Here we focus on the so-called *supermigrativity*, whose definition is given below. In the following, we denote by $\mathbb{I}$ the real unit interval $[0, 1]$.

Definition 1.1. A function $F: \mathbb{I}^2 \to \mathbb{I}$ is called *supermigrative* if, and only if

- it is symmetric, i.e. $F(x, y) = F(y, x)$ for every $(x, y) \in \mathbb{I}^2$;
- it satisfies the inequality

$$F(\alpha x, y) \geq F(x, \alpha y) \tag{1}$$

for all $\alpha \in \mathbb{I}$ and for all $x, y \in \mathbb{I}$ such that $y \leq x$.

When the inequality (1) is strict for any $\alpha \in]0, 1[$ and for all $0 < y < x$, we refer to *strict supermigrativity*.

The term "supermigrative" was used in [12] to underline the connection of this inequality with the concept of *migrativity* of triangular norms (t-norms, for short), originally formulated to study the preservation of associativity under convex combinations [15] and, hence, extended in different situations (see [7,9,18,19]). The study of supermigrativity in various classes of bivariate functions was considered in [12,13]. Since then, several investigations in reliability theory have underlined further applications of this concept in the comparison of random vectors (see, for instance, [6,32,36]).

Here, we aim at revisiting some results about supermigrativity for bivariate semi-copulas (section 2) and present the supermigrativity of more general classes of aggregation functions by discussing similarities and differences. Then, we extend the notion of supermigrativity to arbitrary dimensions (section 3) and present some related inequalities that may arise in a natural way when one is interested in providing bounds for the aggregation process once some of the input values are multiplied by a given rescaling factor (section 4).

2. Supermigrativity of binary semi-copulas and aggregation functions

In this section, we will devote the symbol S to an arbitrary 2-semi-copula, i.e. a binary aggregation function with neutral element 1 (see, e.g., [17]). Note that every semi-copula S satisfies $S(x, 0) = S(0, x) = 0$ for every $x \in \mathbb{I}$ and, hence, Eq. (1) can be considered only for $\alpha \in]0, 1[$.

Given a supermigrative semi-copula S, it follows from Eq. (1), with $x = 1$, that S pointwise dominates the product t-norm (in symbols, $S \geq \Pi_2$, where $\Pi_d(\mathbf{x}) = \Pi_d(x_1, x_2, \ldots, x_d) = x_1 \cdot x_2 \cdots x_d$ for any $d \in \mathbb{N}$). By borrowing a terminology from copula theory (see, for instance, [17]), a binary aggregation function A is said to be PQD (respectively, NQD) when $A \geq \Pi_2$ (respectively, $A \leq \Pi_2$). Thus, any supermigrative semi-copula is PQD.

Remark 2.1. We emphasize that, unlike supermigrative semi-copulas, a supermigrative aggregation function need not be PQD. Consider, for instance, the aggregation function given by $A(x, y) = xy^2$, if $y \leq x$, while $A(x, y) = x^2 y$, otherwise. Thus, A is strictly supermigrative and, at the same time, $A(x, y) < \Pi_2(x, y)$ for all $x, y \in]0, 1[$.

The supermigrativity of Definition 1.1 can be reformulated in various equivalent ways, as indicated in [12, Proposition 2.7] (see also [30]). First, for any dimension $d \geq 2$ we set $\Delta_d := \{\mathbf{x} \in \mathbb{I}^d : 0 < x_d \leq x_{d-1} \leq \cdots \leq x_1\}$.

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