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Characterizations of idempotent discrete uninorms

Miguel Couceiro [a], Jimmy Devillet [b], Jean-Luc Marichal [b]

[a] *LORIA, CNRS, Inria Nancy Grand Est, Université de Lorraine, BP239, 54506 Vandoeuvre-lès-Nancy, France*
[b] *Mathematics Research Unit, University of Luxembourg, Maison du Nombre, 6, avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg*

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Abstract

In this paper we provide an axiomatic characterization of the idempotent discrete uninorms by means of three conditions only: conservativeness, symmetry, and nondecreasing monotonicity. We also provide an alternative characterization involving the bisymmetry property. Finally, we provide a graphical characterization of these operations in terms of their contour plots, and we mention a few open questions for further research.

Keywords: Discrete uninorm; Idempotency; Conservativeness; Bisymmetry; Contour plot; Axiomatization

1. Introduction

Aggregation functions defined on linguistic scales (i.e., finite chains) have been intensively investigated for about two decades; see, e.g., [7–9,11,13–18,21,23]. Among these functions, discrete fuzzy connectives (such as discrete uninorms) are binary operations that play an important role in fuzzy logic.

This short paper focuses on characterizations of the class of idempotent discrete uninorms. Recall that a discrete uninorm is a binary operation on a finite chain that is associative, symmetric, nondecreasing (in each variable), and has a neutral element.

A first characterization of the class of idempotent discrete uninorms was given by De Baets et al. [7]. This characterization reveals that any idempotent discrete uninorm is a combination of the minimum and maximum operations. In particular, such an operation is *conservative* in the sense that it always outputs one of the input values.

The outline of this paper is as follows. After presenting some preliminary results on conservative operations in Section 2, we show in Section 3 that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing (Theorem 12). This new axiomatic characterization is rather surprising since it requires neither associativity nor the existence of a neutral element. We also present a graphical characterization of these operations in terms of their contour plots (Theorem 15) as well as an algebraic translation of this characterization (Theorem 17). These characterizations show us a very easy way to generate all the possible idempotent discrete

E-mail addresses: miguel.couceiro@inria.fr (M. Couceiro), jimmy.devillet@uni.lu (J. Devillet), jean-luc.marichal@uni.lu (J.-L. Marichal).

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uninorms on a given finite chain. In Section 4 we provide an alternative axiomatic characterization of this class in terms of the bisymmetry property. Specifically, we show that the idempotent discrete uninorms are exactly those operations that are idempotent, bisymmetric, nondecreasing, and have neutral elements. More generally, we also show that the whole class of discrete uninorms can also be axiomatized by simply suppressing idempotency in the latter characterization and that this result also holds on arbitrary chains (Theorem 23). Finally, Section 5 is devoted to some concluding remarks and open questions.

This paper is an extended version of a conference paper that appeared in [5].

2. Preliminaries

In this section we present some basic definitions and preliminary results.

Let X be an arbitrary nonempty set and let $\Delta_X = \{(x,x) \mid x \in X\}$.

Definition 1. An operation $F\colon X^2 \to X$ is said to be

- *idempotent* if $F(x,x) = x$ for all $x \in X$;
- *conservative* (or *selective*) if $F(x,y) \in \{x,y\}$ for all $x,y \in X$;
- *associative* if $F(F(x,y),z) = F(x,F(y,z))$ for all $x,y,z \in X$.

Remark 1. Conservative operations were introduced first in [20] and then independently as *locally internal* operations in [12]. By definition, the output value of such an operation must always be one of the input values. In particular, any conservative operation is idempotent. Moreover, such an operation $F\colon X^2 \to X$ can be "discreticized" in the sense that, for any nonempty subset S of X, the restriction of F to S^2 ranges in S. More precisely, it can be shown [4] that the following conditions are equivalent:

(i) F is conservative.
(ii) For any $\varnothing \neq S \subseteq X$, we have $F(S^2) \subseteq S$.
(iii) For any $\varnothing \neq S \subseteq X$ and any $x,y \in S$, if $F(x,y) \in S$ then $x \in S$ or $y \in S$.

Definition 2. Let $F\colon X^2 \to X$ be an operation.

- An element $e \in X$ is said to be a *neutral element* of F (or simply a *neutral element*) if $F(x,e) = F(e,x) = x$ for all $x \in X$. In this case we easily show by contradiction that such a neutral element is unique.
- The points (x,y) and (u,v) of X^2 are said to be *connected for* F (or simply *connected*) if $F(x,y) = F(u,v)$. We observe that "being connected" is an equivalence relation. The point (x,y) of X^2 is said to be *isolated for* F (or simply *isolated*) if it is not connected to another point in X^2.

Proposition 3. *Let $F\colon X^2 \to X$ be an idempotent operation. If the point $(x,y) \in X^2$ is isolated, then it lies on Δ_X, that is, $x = y$.*

Proof. Let $(x,y) \in X^2$ be an isolated point. By idempotency we immediately have $F(x,y) = F(F(x,y),F(x,y))$. But since (x,y) is isolated we necessarily have $(x,y) = (F(x,y),F(x,y))$, and therefore $x = y$. □

Remark 2. We observe that idempotency is necessary in Proposition 3. Indeed, consider the operation $F\colon X^2 \to X$, where $X = \{a,b\}$, defined as $F(x,y) = a$, if $(x,y) = (a,b)$, and $F(x,y) = b$, otherwise. Then (a,b) is isolated and $a \neq b$. The contour plot of F is represented in Fig. 1. Here and throughout, connected points are joined by edges. To keep the figures simple we sometimes omit the edges obtained by transitivity.

We note that any conservative operation $F\colon X^2 \to X$ has at most one isolated point. This observation immediately follows from Proposition 3 and the fact that $F(x,y) \in \{x,y\} = \{F(x,x),F(y,y)\}$ for all $x,y \in X$.