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Equivalence relations induced by the U-partial order ☆

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Abstract

In this paper, an equivalence relation on the class of uninorms with the same neutral element induced by a U-partial order is provided and discussed. The equivalence classes linked to some special uninorms are characterized. Furthermore, the set of incomparable elements with respect to the U-partial order is deeply investigated. Finally, another equivalence relation on the class of uninorms with the same neutral element is introduced and studied.

Keywords: Uninorm; Equivalence class; U-partial order; Distributivity

1. Introduction

In [20], Yager and Rybalov introduced uninorms on the unit interval [0, 1], which are a special kind of aggregation functions that generalize both t-norms and t-conorms (see [5,6,11,15]), with more applications like fuzzy logic framework (see [17]), fuzzy quantifiers (see [17]), neural networks (see [18]), experts systems (see [18]), fuzzy system modeling (see [7,21]), and so on. Due to this great number of applications, uninorms have also been studied from the purely theoretical point of view. In this way, one of the main topics is directed towards the characterization of those uninorms that verify certain properties that may be useful in each context. Recently, the order generating problem from logical operators have been considered by many researchers (see [2,8,12,14,16]).

In [14], a partial order defined by means of t-norms on a bounded lattice has been introduced. Let L be a bounded lattice, and T a t-norm on L. Define a partial order $\preceq_T$ as follows:

$$\forall\, x,\ y \in L,\ x \preceq_T y \text{ if and only if } T(l, y) = x \text{ for some } l \in L.$$

This partial order $\preceq_T$ is called a T-partial order on L. Based on T-partial orders, Kesicioğlu, Karaçal and Mesiar (see [16]) introduced an equivalence relation on the class of t-norms on a bounded lattice, and characterized the equivalence classes linked to some special t-norms. It is well known that the structure of uninorms is a special combination of

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t-norms and t-conorms. In [8], Ertuğrul, Kesicioğlu and Karaçal defined a partial order $\preceq_U$ induced by a uninorm U on a bounded lattice L, which extends the T-partial order to a more general form. They also investigated some connections between the natural order $\leq$ on L and the partial order $\preceq_U$. In this paper, we call the partial order $\preceq_U$ a U-partial order on L.

There are many special uninorms. For instance, in [9], there are two special uninorms $\underline{U}_e$ and $\overline{U}_e$ on [0, 1], which are the weakest uninorm and the strongest uninorm, where e is a neutral element in [0, 1]. Also, there exist representable uninorms, idempotent uninorms, uninorms with given continuous underlying t-norms and t-conorms, uninorms which are continuous on $(0, 1)^2$, and uninorms with N-dual underlying operators, where N is a strong negation with $N(e) = e$. Furthermore, for any uninorm U with neutral element e, $\phi : [0, 1]^2 \longrightarrow [0, 1]$ an increasing bijection with $\phi(e) = e$, we can obtain a new uninorm $U_\phi : [0, 1]^2 \longrightarrow [0, 1]$ as follows: $U_\phi(x, y) = \phi^{-1}(U(\phi(x), \phi(y)))$ for all $x, y \in [0, 1]$. Motivated by the work in [16], we define an equivalence relation $\sim$ on the class of uninorms with the same neutral element induced by the order $\preceq_U$. The main purpose of this paper is to characterize the equivalence classes obtained from ordering $\preceq_U$ of these special uninorms.

Distributivity between two operations is a property that was already posed many years ago and that is especially important in the framework of logical connectives (see [1]). In [19], the authors gave a detailed study of the distributive equation between two uninorms U_1 and U_2

$$U_1(x, U_2(y, z)) = U_2(U_1(x, y), U_1(x, z)), \forall\, x, y, z \in [0, 1].$$

We study the relationship between distributivity and equivalence classes based on the above equivalence relation. Moreover, let K_U denote the set of all elements from [0, 1] admitting some incomparability with respect to $\preceq_U$. We study the set K_U in detail. Finally, we introduce another equivalence relation $\sim_k$ on the class of uninorms with the same neutral element induced by K_U, and prove that $\sim \subsetneqq \sim_k$.

2. Preliminaries

In this section, we shall just give some basic facts about uninorms.

Definition 2.1. ([13]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \longrightarrow L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

A uninorm U is called a t-norm (t-conorm, respectively), if it has a neutral element $e = 1$ ($e = 0$, respectively).

Example 2.2. ([15]) Define a map $T_D : [0, 1]^2 \longrightarrow [0, 1]$ as follows:

$$\forall\, x, y \in [0, 1],\ T_D(x, y) = \begin{cases} 0, & (x, y) \in [0, 1)^2, \\ \min(x, y), & \text{otherwise,} \end{cases}$$

and define a map $S_D : [0, 1]^2 \longrightarrow [0, 1]$ as follows:

$$\forall\, x, y \in [0, 1],\ S_D(x, y) = \begin{cases} 1, & (x, y) \in (0, 1]^2, \\ \max(x, y), & \text{otherwise.} \end{cases}$$

Then T_D is a t-norm, and S_D is a t-conorm. Moreover, T_D is the smallest t-norm and S_D is the greatest t-conorm.

Proposition 2.3. *([9]) Let $U \in \mathcal{U}(e)$ with $e \in (0, 1)$. Then the function $T_U : [0, 1]^2 \longrightarrow [0, 1]$ defined by*

$$\forall\, x, y \in [0, 1],\ T_U(x, y) := \frac{U(ex, ey)}{e}$$

is a t-norm, and the function $S_U : [0, 1]^2 \longrightarrow [0, 1]$ defined by

$$\forall\, x, y \in [0, 1],\ S_U(x, y) := \frac{U(e+(1-e)x,\ e+(1-e)y)-e}{1-e}$$

is a t-conorm.