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Short communication

A note on "Distributivity and conditional distributivity of a uninorm with continuous underlying operators over a continuous t-conorms"

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Abstract

In this paper, the study of the distributivity equation involving uninorms and t-conorms given in Li and Liu (2016) [3] is revised. Some errors in the proof of theorems in the mentioned reference are pointed out and their right versions are offered. Furthermore, the class of uninorms with continuous underlying operators which are (conditionally) distributive over a continuous t-conorm is characterized.

Keywords: Uninorm; Conditional distributivity; Distributivity; Triangular conorm

We assume the reader to be familiar with some basic notions about t-norms and t-conorms which can be found in [2] (including the standard notation for ordinal sum of t-conorms). Also, some fundamental results on uninorms in general and their structure can be found in [1,8].

The structure of some cases of uninorms with continuous underlying operators was studied in Section 3 of [3]. Especially, a full characterization of the class of uninorms with strict underlying operators was stated in Theorem 3 of [3]. Unfortunately, there exists one error in the proof of Theorem 3. For Case 2 in the proof of Theorem 3, in order to prove that $U(1, x) = 1$ or x for all $x \in]0, e[$, we have the following equation (by induction from Eq. (10))

$$c = U(1, c) = U(1, U(c, \ldots\ldots, c))$$

where $c = U(1, t)$, $t \in]0, e[$. Note that the above equation is incorrect which can not be obtained by induction from Eq. (10) from [3]. Indeed, we can only obtain $c = U(1, c)$ by induction from Eq. (10) in [3]. However, we can prove that the above result still holds by Proposition 2, Proposition 3 in [5] or more general result as follows.

Lemma 1. [6, Lemma 2] *Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$. If $a \in [0, 1]$ is an idempotent element of U, then $U(a, x) \in \{a, x\}$ for all $x \in [0, 1]$.*

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Proof. For the completeness of this paper, we present the proof here. To prove the result, we distinguish three cases.

(i) $a = e$. The result holds obviously.
(ii) $a \in]e, 1]$. Since the underlying t-conorm S_U is continuous, $U(a, x) = \max(a, x)$ for all $x \in]e, 1]$. On the contrary, suppose that there exists $x_0 \in [0, e[$ such that $U(a, x_0) = b \in]x_0, a[$. Then $U(a, b) = U(a, U(a, x_0)) = U(U(a, a), x_0) = U(a, x_0) = b$ by the associativity of U. If $b \geq e$ then $b = U(a, b) \geq \max(a, b) = a$, a contradiction. Thus $b < e < a$. Since the underlying t-norm T_U is continuous, there exists $x_1 \in [0, e]$ such that $U(b, x_1) = x_0$. Then, we have the following contradiction

$$U(a, x_0) = U(a, U(b, x_1)) = U(U(a, b), x_1) = U(b, x_1) = x_0.$$

(iii) $a \in [0, e[$. The proof is analogical with case (ii). ⋄

Remark 1. (i) Due to 1 is an idempotent element of uninorm U, we have $U(1, x) = 1$ or x for all $x \in]0, e[$ by Lemma 1. Hence, the result holds in Case 2 of Theorem 3 in [3]. Note that the other parts in the proof of Theorem 3 in [3] are correct.

(ii) For Case 1 in the proof of Theorem 4 in [3], there exists a similar error. Furthermore, a similar error appears in the proof of Theorem 3 in [4]. These errors can also be corrected by Lemma 1.

In [3], we have investigated the distributivity and conditional distributivity of a uninorm U with continuous underlying operators over a continuous t-conorm S, i.e., the following two equations

$$U(x, S(y, z)) = S(U(x, y), U(x, z)) \text{ for all } x, y, z \in [0, 1]. \tag{1}$$

$$U(x, S(y, z)) = S(U(x, y), U(x, z)) \text{ whenever } x, y, z \in [0, 1], S(y, z) < 1. \tag{2}$$

Proposition 1. *[3, Proposition 2] Let U be a uninorm with neutral element $e \in]0, 1[$ and S be a t-conorm fulfilling $S(e, e) = e$. Then U and S satisfy Eq. (2) if and only if $S = S_M$.*

Lemma 2. *[3, Lemma 3] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and S be a continuous t-conorm fulfilling $S(e, e) > e$. If U and S satisfy Eq. (2), then S has only one ordinal summand.*

Theorem 1. *[3, Theorem 8] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and $S = (\langle a, b, S^* \rangle)$ be a continuous t-conorm. If U and S satisfy Eq. (2), then the set $[a, b]^2$ is closed under U.*

Theorem 2. *[3, Theorem 9] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and S be a strict t-conorm. Then U and S satisfy Eq. (2) if and only if U is a representable uninorm which multiplicative generator is also an additive generator of S.*

Theorem 3. *[3, Theorem 10] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and S be a nilpotent t-conorm. Then U and S do not satisfy Eq. (2).*

Proposition 2. *[3, Proposition 3] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and $S = (\langle a, b, S^* \rangle)$ be a continuous t-conorm, $a > 0$ or $b < 1$. If U and S satisfy Eq. (2), then the following statements hold.*

(i) $a = \max\{z \in [0, e[: U(z, z) = z\}$.
(ii) $b = \min\{z \in]e, 1] : U(z, z) = z\}$.

Proposition 3. *[3, Proposition 4] Let U be a uninorm with continuous underlying operators and neutral element $e \in]0, 1[$ and $S = (\langle a, b, S^* \rangle)$ be a continuous t-conorm, $a > 0$ or $b < 1$. If U and S satisfy Eq. (2), then the following statements hold.*

(i) *There exists a representable uninorm R such that $U(x, y) = a + (b - a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $(x, y) \in [a, b]^2$ and S^* is strict.*