Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 333 (2018) 87–105

FUZZY sets and systems

www.elsevier.com/locate/fss

Fuzzy Turing machines: Normal form and limitative theorems

Giangiacomo Gerla

Department of Mathematics University of Salerno, via Giovanni Paolo II, I-84084 Fisciano, SA, Italy

Received 21 February 2016; received in revised form 16 December 2016; accepted 18 January 2017
Available online 2 February 2017

Abstract

A normal form for fuzzy Turing machines is proposed and examined. This normal form is arithmetical in nature since the truth values are substituted by n-ples of natural numbers and the operation interpreting the conjunction becomes a sort of truncated sum. Also, some of the results in the paper enable us to emphasize the inadequacy of the notion of fuzzy Turing machine for fuzzy computability, i.e. that this notion is not a good candidate for a 'Church thesis' in the fuzzy mathematics framework.

Keywords: Fuzzy Turing machines; Fuzzy computability; Normal form; Church thesis

1. Introduction

As it is well known, the notions of '*semidecidability*', '*decidability*' and '*computability*' are basic for classical logic, computability theory, formal language theory [15]. For example, in absence of these notions almost all the results in classical formal logic lose their meaning. Indeed, a completeness theorem could be achieved simply by assuming as a system of logical axioms the whole set of logically true sentences. Also, it would be possible to obtain the axiomatizability of arithmetic by choosing as a system of proper axioms the whole set of formulas true in the canonical model. Now, exactly the same observations hold true for formal fuzzy logic and this makes crucial the definition of an adequate theory of computability in the fuzzy framework. This is particularly true for Pavelka approach to fuzzy logic [2] since it should be natural to expect from an inferential apparatus the semidecidability of the fuzzy subset of theorems of a decidable fuzzy theory.

Now, since the whole (classical) computability theory is based on the notion of Turing machine, it is not surprising that in literature definitions of fuzzy Turing machine are proposed and investigated (see [22,1,16,18,20,21]). These definitions require the use of algebraic structures to evaluate the acceptability degree of the transition rules of a fuzzy Turing machine and to calculate the truth values of the relative accepted fuzzy language.

The aim of this paper is to show that every fuzzy Turing machine whose set of truth values is totally ordered is reducible to a machine based on a numerical monoid, i.e. a totally ordered commutative monoid whose elements are (classes of) p-ples of natural numbers. Also, some of the results in the paper enable us to emphasize the inadequacy

E-mail address: gerla@unisa.it.

http://dx.doi.org/10.1016/j.fss.2017.01.008
0165-0114/

of the notion of fuzzy Turing machine for a 'Church thesis' in the fuzzy framework. All the results and considerations in the paper hold true also for the fuzzy grammars and for all the notions in fuzzy logic based on totally ordered commutative monoids (for example, fuzzy control and fuzzy logic programming).

Notice that all the theorems related to the ordered semigroups in this paper are either easy to prove or well known in specialized literature (see for example [4]). Indeed, this paper is merely an application to fuzzy logic of the powerful and elegant theory of the ordered semigroups.

2. Partially ordered monoids for the fuzzy Turing machines

We denote by N the set of natural numbers different from zero and we put

$$N_0 = N \cup \{0\}, \qquad N_\infty = N \cup \{\infty\}, \qquad N_{0,\infty} = N \cup \{0, \infty\}.$$

The usual addition and order are extended to $N_{0,\infty}$ by assuming that $\infty + x = x + \infty = \infty$ and that ∞ is the maximum. A *type* is a p-ple $(o(1), \dots, o(p))$ of elements in N_∞. We denote by $\Re$ the set of real numbers and by [0, 1] the interval $\{x \in \Re : 0 \leq x \leq 1\}$. Given an ordered set $\boldsymbol{A} = (A, \leq, 0, 1)$ in which 0 is the minimum and 1 is the maximum, we denote by $x \wedge y$ and $x \vee y$ the last upper bound and the greater upper bound of the set $\{x, y\}$ (in the case they exist). We call *fuzzy subset* of a nonempty set S a map $s : S \to A$ from S to A (for elementary notions in fuzzy set theory, see for example [9,12–14]). We put $values(s) = \{s(x) : x \in S\} - \{0, 1\}$. An *alphabet* is a finite nonempty set I, a (*formal*) *language* is a subset of the set I^+ of words on I, a *fuzzy language* is a fuzzy subset of I^+.

In fuzzy logic the domain of $\boldsymbol{A}$ is interpreted as the set of truth values and it is equipped with suitable operations to interpret all the logical connectives. Now, there are several important notions in fuzzy set theory using only an operation for the conjunction and the join operator. Examples are furnished by the fuzzy Turing machines and the fuzzy grammars. Further examples are in fuzzy control and in fuzzy logic programming. This suggests considering a particular class of complete, partially ordered monoid.

Definition 2.1. A *pomonoid* (*tomonoid*) is a structure $\boldsymbol{A} = (A, \cdot, \leq, 1)$ such that

- $(A, \cdot, 1)$ is a monoid
- $\leq$ is a partial (total) order
- the operation $\cdot$ is order-preserving, i.e. for every x, y, z in A, $x \leq y$ entails $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$.

A structure $\boldsymbol{A} = (A, \cdot, \leq, 0, 1)$ is a *conjunction monoid* (*tomonoid*) if

- $\boldsymbol{A} = (A, \cdot, \leq, 1)$ is a commutative pomonoid (tomonoid);
- $0 \neq 1$, 0 is the minimum and 1 is the maximum in $(A, \leq)$.

Notice that, since 1 is the identity, $0 \cdot 1 = 0$ and therefore, since $\leq$ is order-preserving and 1 is the maximum, $0 \cdot x \leq 0 \cdot 1 = 0$ for every x in A. This proves that $0 \cdot x = 0$. The expression 'conjunction monoid' is suggested by the fact that in a multi-valued logic the properties listed in Definition 2.1 are the usual ones satisfied by the operation used to interpret the conjunction. We adopt the multiplicative notation since this is in accordance with the tradition in multi-valued logic, obviously there is no difficulty to adopt an additive notation.

The main examples of conjunction tomonoid are furnished by the continuous triangular norms (see for example [11]).

Definition 2.2. A *continuous triangular norm* is a continuous binary operation in the interval [0, 1] which is associative, commutative and monotone and whose identity element is 1. A *T-conjunction monoid* is a conjunction monoid whose operation is a continuous triangular norm.

The following are three basic examples of a T-conjunction monoid:

- $\boldsymbol{A}$ is the *Gödel monoid* if $\cdot$ is the minimum,
- $\boldsymbol{A}$ is the *Łukasiewicz monoid* if $\cdot$ is defined by setting $x \cdot y = \max\{x + y - 1, 0\}$,