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Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata ☆

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Abstract

In this paper we consider different types of ranks of fuzzy matrices over residuated lattices. We investigate relations between ranks and prove that row rank, column rank and Schein rank of idempotent fuzzy matrices are equal. In particular, ranks and corresponding decompositions of fuzzy matrices representing fuzzy quasi-orders are studied in detail. We show that fuzzy matrix decomposition by ranks can be used in the state reduction of fuzzy automata. Moreover, we prove that using rank decomposition of fuzzy matrices improves results of any state reduction method based on merging indistinguishable states of fuzzy automata.

Keywords: Fuzzy matrices; Fuzzy matrix decomposition; Matrix subdecomposition; Fuzzy quasi-order; State reduction; Residuated lattice

1. Introduction

The development of fuzzy matrix theory started in 1971, when Zadeh introduced fuzzy equivalence relations [32]. Fuzzy matrices emerged as a mean of representing fuzzy equivalences and fuzzy relations between finite sets in general. Since then, together with fuzzy relations, fuzzy matrices appeared to be useful in many different contexts such as: fuzzy control, approximate reasoning, fuzzy cluster analysis, fuzzy neural networks, fuzzy decision making, fuzzy cognitive and fuzzy relational mapping, etc. However, independently of fuzzy relations, development of a fuzzy matrix theory began in 1980, when K. H. Kim and F. W. Roush introduced a general framework of the theory of fuzzy matrices as a generalization of the Boolean matrix theory and the theories of matrices over nonnegative real numbers, nonnegative integers, and over other types of semirings. They established basic concepts of linear algebra over fuzzy matrices: spaces of fuzzy vectors, row and column basis, Schein rank, row and column rank, regular fuzzy matrices, eigenvectors and eigenvalues, etc. Ever since, fuzzy matrices appeared to be the topic of many studies in different areas of mathematics: computation of generalized inverses of regular fuzzy matrices, as well as solvability

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and solution computation of linear fuzzy matrix equations and other classical problems in linear algebra [1–3,15,16,24–26,29].

In most of the aforementioned papers, fuzzy matrices over the fuzzy algebra (also known as the Gödel structure) and over chain semirings were studied, whereas in the last ten years the researchers focused on fuzzy relations, and consequently on fuzzy matrices over more general structures – residuated lattices. In 1999, Bělohlávek started his work on formal concept analysis by introducing fuzzy Galois connections as a generalization of Galois connections from the point of view of fuzzy logic with membership values in residuated lattices [4]. Later, in [5] fixed points of fuzzy Galois connections were proven to be special cases of *maximal subdecompositions* of fuzzy matrices. Through a series of papers, while studying formal concept analysis, Bělohlávek and his coworkers provided significant results regarding properties of row and column spaces associated to fuzzy matrices [11,12], optimal decompositions of fuzzy matrices [8,9], and efficient computation of suboptimal decompositions of fuzzy matrices [14]. It is worth noting that Ćirić et al. [17–21,23,30,31] made a great contribution to the theory of fuzzy matrices over complete residuated lattices, and offered a comprehensive survey on the theory of fuzzy automata, particularly relevant to problems of state reduction, determinization and of bisimulation of fuzzy automata.

Basic notions and notation concerning subdecompositions and decompositions, row and column spaces, Schein ranks, and row and column ranks of fuzzy matrices over residuated lattices are introduced in Section 2. In Section 3, we investigate and enclose several properties of fuzzy matrices having equal column (row) spaces. We also offer some characterizations of ranks of fuzzy matrices and prove that for a given fuzzy matrix A and any of its ranks, there exists a corresponding decomposition. By using these results, we prove that all ranks of idempotent fuzzy matrices are equal, which lead us to the same result regarding fuzzy-quasi order matrix. It is well known (cf. [22,31]) that the cardinality of a set of columns of a fuzzy quasi-order matrix Q is equal to the cardinality of the set of rows of Q, and that number we denote $d(Q)$. We first show that the rank of fuzzy quasi-order matrices can be strictly smaller than $d(Q)$. Therefore, $d(Q)$ is a "good" candidate for estimating the rank of fuzzy quasi-order matrix Q, in the sense that it is efficiently computable and the corresponding $d(Q)$-decomposition of Q is also efficiently computable.

In Section 4, we continue to investigate ranks and various types of subdecompositions of fuzzy quasi-order matrices over residuated lattices. We first introduce maximal subdecompositions and maximal decompositions of matrices as the main tool for our further research. After examining basic properties of maximal subdecompositions and maximal decompositions, we prove that the cardinality of any set of columns (resp. rows) of a fuzzy quasi-order matrix Q, which is the spanning set of the column space $\mathscr{C}(Q)$ (resp. row space $\mathscr{R}(Q)$), is also good for estimation of its rank. We provide a procedure for testing whether a set of rows (resp. columns) of a fuzzy quasi-order matrix is the spanning set for the column space (resp. row space) of that matrix. That lead us to the conclusion that the best candidate of that kind is the cardinality of the spanning set of columns (rows), which is minimal w.r.t. set inclusion. However, we give an example of a fuzzy quasi-order matrix Q with many different minimal sets of columns (resp. rows) which are spanning sets of the column space (resp. row space) and have different sizes. Ultimately, we give the main result of this section – characterization of residuated lattices over which any fuzzy quasi-order matrix Q has rank $\rho(Q)$ equal to $d(Q)$. It is worth noting that all of the results in this section are proven using maximal subdecompositions and maximal decompositions of fuzzy matrices, and since they are both finite suprema of certain formal concepts, tools and results of Bělohlávek and his coworkers are intensely used.

In Section 5, we explain how decomposition of fuzzy matrices can improve results of state reduction methods based on merging undistinguishable states of fuzzy automata. The basic idea of reducing the number of states of non-deterministic automata is to compute and merge indistinguishable states. It resembles the minimization algorithm for deterministic automata, but is more complicated. In the fuzzy case, indistinguishability is modeled by crisp equivalences [28,30], fuzzy equivalences [17,18], and fuzzy quasi-orders [30,31]. In all the aforementioned papers matrices of fuzzy relations used in state reductions are fuzzy quasi-order matrices and are solutions to the *general system*. Moreover, the state reduction of a fuzzy automaton using a fuzzy quasi-order matrix Q produces a fuzzy automaton that is equivalent to the starting automaton and its number of states is equal to $d(Q)$. Our improvement is not based on computing solutions to the general system, but on using already computed solutions in a better way. Namely, if Q is a solution to the general system, we compute a k-decomposition (L, R) of Q and make a new fuzzy automaton called (L, R)-*transformation* of the starting fuzzy automaton. An (L, R)-transformation of a fuzzy automaton is equivalent to starting one and has k states. In the case when Q is a fuzzy quasi-order and $k < d(Q)$, provided (L, R)-transformation is efficiently computable, our method yields an improvement.

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