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Some characterisations of self-dual aggregation functions when relative shortfalls are considered

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Abstract

Whenever shortfalls are defined as the absolute difference between the upper bound and the level of attainments the characterisation of aggregation functions that rank attainment and shortfall distributions mirroring one another, i.e. self-dual aggregation functions, is a widely discussed issue. In this paper we consider an alternative definition of shortfalls as the relative difference between the upper bound and the level of attainments and extend some characterisation results to this new framework. Moreover, we propose a particular dual decomposition for each aggregation function and apply it to two major classes of homogeneous aggregation functions: α-power means and OWA operators.

Keywords: Self-duality; Homogeneity; Aggregation functions; Dual decomposition

1. Introduction

The distribution of a bounded variable can be described in terms either of attainment or shortfall. The classical aggregation functions applied to these distributions appear to be one-sided, i.e. rankings of aggregates by attainment and shortfall do not necessarily mirror one another.[2] An obvious consequence is that there may be huge differences between the two approaches. As some researchers have stated, it is possible to find aggregation functions such that the aggregate value of the shortfall distribution coincides with the shortfall of the aggregate value of the original attainment distribution (see among others Calvo et al. [6], García-Lapresta and Marques Pereira [11,12] and García-Lapresta et al. [10]). These are said to be self-dual aggregation functions. Aggregation functions are not in general self-dual, but Calvo et al. [6] and García-Lapresta and Marques Pereira [11] propose methods by which self-dual

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2 A similar problem is tackled in the inequality context by, among others, Erreygers [8], Lambert and Zheng [16], Lasso de la Vega and Aristondo [17] and Aristondo et al. [3].

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aggregation functions can be associated with any aggregation function. In such papers shortfalls are defined as the absolute differences between the upper bound and the level of attainment. Given the predominance of the relative approach in the literature it would be interesting to see if analogous results can be found when shortfalls are defined as the relative differences between the upper bound and the level of attainment. What is implicitly behind each definition of "shortfall" is a different idea of distance, i.e. of the numerical description of how far apart the upper bound and each attainment level are. Choosing one or another may influence how big each shortfall is, and researchers need to be alert to the distinction and choose the definition that best fits their purposes. For some purposes it would therefore be of interest to find aggregation functions for which the aggregate value of the shortfalls coincides with the shortfall of the aggregate value of the original attainments when the idea of distance implicitly assumed is relative. That is the main aim of this paper. We offer two characterisations of self-duality for aggregation functions by considering relative shortfalls, which are the logarithmic transformations of absolute shortfalls. The two characterisations take their cues from the papers by Calvo et al. [6] and García-Lapresta and Marques Pereira [11] respectively. Pointing out that in both characterisations each self-dual function is obtained by a particular way of combining an aggregation function with its dual and following Maes et al. [18], we provide a characterisation of self-dual aggregation functions which preserve homogeneity of degree 1. Moreover, following García-Lapresta and Marques Pereira [11], we introduce a particular dual decomposition for any aggregation function whenever relative shortfalls are considered. This decomposition is applied to two important classes of homogeneous aggregation functions: α-power means and OWA operators, as introduced by Yager [22].

The rest of the paper is organised as follows. Section 2 reviews basic notions regarding aggregation functions and some of their properties. Section 3 presents the new property of self-duality for aggregation functions and some characterisation results. Section 4 proposes a particular dual decomposition for each aggregation function in this new context. Sections 5 and 6 respectively examine the dual decomposition for two classes of homogeneous aggregation functions. Section 7 concludes.

2. Notation and aggregation functions

We assume throughout that variables are drawn from an interval $[a, b]$ which is a subset of $[0, \infty)$. Points in $[a, b]^n$ with $n \in \mathbb{N}$ will be denoted by boldface characters: $\boldsymbol{x} = (x_1, \ldots, x_n)$. For $x \in [a, b]$, we denote $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [a, b]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$; by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [a, b]^n$, with $(x_{(1)}, \ldots, x_{(n)})$ we denote the increasing ordered version of $\boldsymbol{x}$, i.e. $x_{(i)}$ is the i-th lowest number of $\{x_1, \ldots, x_n\}$. Hence, $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$. Given a permutation on $\{1, \ldots, n\}$, i.e. a bijection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, with $\boldsymbol{x}_\sigma$ we denote $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We begin by defining standard properties of real functions on $[a, b]^n$. Readers interested in further are directed to Fodor and Roubens [9, Chapter 5], Calvo et al. [6], Beliakov et al. [4], García-Lapresta and Marques Pereira [11] and Grabisch et al. [13].

Definition 1. Let $A : [a, b]^n \longrightarrow \mathbb{R}$ be a function.

1. A is *idempotent* if for every $x \in [a, b]$:
$$A(x \cdot \mathbf{1}) = x.$$
2. A is *symmetric* if for every permutation σ on $\{1, \ldots, n\}$ and every $\boldsymbol{x} \in [a, b]^n$:
$$A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x}).$$
3. A is *monotonically increasing* if for all $\boldsymbol{x}, \boldsymbol{y} \in [a, b]^n$:
$$\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$
4. A is *strictly monotone increasing* if for all $\boldsymbol{x}, \boldsymbol{y} \in [a, b]^n$:
$$\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y}).$$
5. A is *monotonically decreasing* if for all $\boldsymbol{x}, \boldsymbol{y} \in [a, b]^n$:
$$\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \leq A(\boldsymbol{y}).$$