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Dualities in the class of extended Boolean functions

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Abstract

We introduce and discuss duality operators on the set of binary extended Boolean functions, i.e., on the set of binary operations on the real interval $[0, 1]$ whose restrictions to Boolean inputs yield Boolean functions. These dualities have been divided into seven classes, and the majority of their properties depend on the class they belong to. We study composition of dualities, properties of dual classes, duality of properties of extended Boolean functions and invariantness of extended Boolean functions with respect to particular dualities. Our approach allows to transfer the results known for some studied class of extended Boolean functions into the results for the corresponding dual classes. As typical examples, one can recall the standard duality of the classes of all t-norms and t-conorms and the duality of implication functions and conjunctive (disjunctive) aggregation functions.

Keywords: Aggregation function; Boolean function; Duality; Extended Boolean function; Implication function; t-norm; t-conorm

1. Introduction

Recall that binary Boolean functions are mappings $B\colon \{0,1\}^2 \to \{0,1\}$, and that they appear in many basic mathematical domains, especially in logic. There are exactly $2^4 = 16$ binary Boolean functions. The set containing all of them will be denoted by $\mathscr{B}$. We can write $\mathscr{B} = \{B_i \mid i \in \{0, \dots, 15\}\}$, where $B_i(0,0) = i_0$, $B_i(1,0) = i_1$, $B_i(0,1) = i_2$, $B_i(1,1) = i_3$, and (i_3, i_2, i_1, i_0) is a transcription of i in the $\{0,1\}$-base, i.e., $i = \sum_{j=0}^{3} i_j 2^j$. So, for example, B_8 is the

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standard Boolean conjunction, B_{14} is the Boolean disjunction, B_{13} is the Boolean implication, while B_9 is the Boolean biimplication.

Since Kolmogorov's axiomatization of probability theory and Zadeh's proposal of fuzzy sets, many sources of information have dealt with values from the real unit interval [0, 1] as quantitative inputs and several types of binary operations on [0, 1] extending Boolean operations have been considered for the fusion of input information. In this paper we will deal with particular operations that assign to any two inputs from [0, 1] an output from [0, 1] and extend the Boolean functions from $\mathscr{B}$.

Definition 1. A function $F\colon [0,1]^2 \to [0,1]$ extending some binary Boolean function, i.e., satisfying the property $F(0,0), F(1,0), F(0,1), F(1,1) \in \{0,1\}$, will be called an extended Boolean function. The set of all binary extended Boolean functions will be denoted by $\mathscr{F}$.

We introduce (binary) extended Boolean functions (EB-functions for short) in order to build a common general concept for some particular well-known classes of functions that are frequently applied in real data processing, such as, for example, conjunctive and disjunctive aggregation functions [5,18], overlap functions [12], grouping functions [14], restricted equivalence functions [9], symmetric difference operators [24], fuzzy implications [2], coimplicators [3,23], fuzzy bi-implication functions [17], etc. This enables us to study many problems in general framework of all (binary) EB-functions $\mathscr{F}$ and formulate consequences for particular subclasses of EB-functions. In this paper, we will study dualities of EB-functions and their subclasses. Though our approach is purely theoretical, it offers a big potential in transferring the results known for some particular class of EB-functions to the corresponding dual classes. This is, for example, the case of implication functions and aggregation functions with annihilator $a = 1$ (disjunctors), or aggregation functions with annihilator $a = 0$ (conjunctors). Several other examples of using dualities can be found in fuzzy logic [19], fuzzy set theory [8,15,16], aggregation theory [4,7,21], image processing [10,11], etc.

Formally, the structure $([0,1], F)$ with an EB-function F can be understood as a particular grupoid, and thus several algebraic results concerning grupoids can be considered. However, having in mind that a functional look at considered operations on [0, 1] is more common in their application fields, we prefer to deal with EB-functions and a functional approach here.

In general, a duality of EB-functions is a non-identity mapping $\mathfrak{D}\colon \mathscr{F} \to \mathscr{F}$ that is involutive, i.e., for each $F \in \mathscr{F}$, $\mathfrak{D}$ satisfies the property $\mathfrak{D}(\mathfrak{D}(F)) = F$. Let us first recall the well-known standard duality of EB-functions that assigns to an EB-function F a dual function F^d, defined by $F^d(x,y) = 1 - F(1-x, 1-y)$. It is clear that for each $F \in \mathscr{F}$, $F^d \in \mathscr{F}$, too, and $\left(F^d\right)^d = F$. Some subclasses of EB-functions (say $\mathscr{F}_1, \mathscr{F}_2$) are dual in the sense that $F \in \mathscr{F}_1$ if and only if $F^d \in \mathscr{F}_2$. This is, for example, the case of the class $\mathscr{T}$ of all t-norms and the class $\mathscr{S}$ of all t-conorms, or of the class Ω of all overlap functions and the class Γ of all grouping functions.

For any strong fuzzy negation N, i.e., a decreasing function $N\colon [0,1] \to [0,1]$ that is involutive (for each $x \in [0,1]$, $N(N(x)) = x$), one can consider an N-duality of EB-functions that maps $F \mapsto F^N$, where

$$F^N\colon [0,1]^2 \to [0,1], \quad F^N(x,y) = N(F(N(x)), N(y))).$$

Clearly, for each $F \in \mathscr{F}$, $F^d = F^{N_s}$, where N_s is the standard (Zadeh) fuzzy negation given by $N_s(x) = 1 - x$, $x \in [0,1]$. Note that we will call strong fuzzy negations simply negations when no confusion can arise.

If we consider a duality $\mathfrak{D}$ of EB-functions, then for any subclass $\mathscr{G} \subset \mathscr{F}$ we define its dual class by $\mathfrak{D}(\mathscr{G}) = \{\mathfrak{D}(F) \mid F \in \mathscr{G}\}$. Clearly, $\mathfrak{D}(\mathfrak{D}(\mathscr{G})) = \mathscr{G}$. It is important to know which property of EB-functions from the class $\mathfrak{D}(\mathscr{G})$ corresponds to a considered property p of EB-functions from the class $\mathscr{G}$. This knowledge enables one to discuss the class $\mathscr{G}$ of EB-functions only, and use the obtained results for deriving the results for the dual class $\mathfrak{D}(\mathscr{G})$. For example, continuity is preserved by any N-duality, and so, the well-known representation of continuous t-norms by means of ordinal sums can be transformed straightforwardly into the representation of continuous t-conorms [20].

The aim of this paper is to bring a deeper study of dualities defined on EB-functions. The paper is organized as follows. The next section provides definitions of important notions exploited throughout the paper. In Section 3 we define and study D-dualities of EB-functions and composition of dualities. Section 4 is devoted to the study of properties of dual classes, and in Section 5 we focus on the study of duality of properties of EB-functions. Finally, in Section 6 we discuss the invariantness of EB-functions with respect to particular dualities. The last section contains several additional remarks concerning the studied topic. In particular, we emphasize the important fact that our results

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