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A parametrized sum of fuzzy numbers with applications to fuzzy initial value problems

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Abstract

This paper presents a new parametrized family of joint possibility distributions that can be used to control the Pompeiu–Hausdorff norm of the sum resulting from the extension principle. We are able to generate sums with norms covering the entire interval ranging from the smallest possible value to the largest one. Finally, we apply our approach to numerical methods for solving first-order fuzzy initial value problems.

Keywords: Addition of fuzzy numbers; Interactive fuzzy numbers; Completely correlated fuzzy numbers; sup-J extension principle; Fuzzy initial value problems

1. Introduction

In many areas such as economy, finance, medicine, biology, chemistry, etc., one can observe phenomena of interactivity when combining uncertain quantities described by fuzzy numbers using arithmetic operations.

In many real-life situations, one quantity influences the other to a certain, often unknown degree, thereby affecting the results produced by an arithmetic operation. When expressing the quantities under consideration as fuzzy numbers, this interaction can be described using the concept of interactivity of fuzzy numbers.

This notion can be defined in terms of a joint possibility distribution [1], which Carlsson et al. [2] employed to propose a generalization of Zadeh's extension principle [3] called sup-C extension principle. In a recent paper, Coroianu speaks of the interactive extension with respect to a joint possibility distribution C instead of the sup-C extension principle [4]. Since that the term "sup-C" stands for a type of fuzzy relational composition in the literature of fuzzy set theory, we prefer to speak of the sup-J extension principle instead of the sup-C extension principle.

This paper focuses on the addition of fuzzy numbers using the sup-J extension principle. The properties concerning the addition of interactive fuzzy numbers have been investigated by many researchers [4–9]. For example, Coroianu

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established conditions for the addition by means of sup-J extension principle to coincides with the usual sum of fuzzy numbers, that is, the sum based on Zadeh's extension principle [4].

In this paper, we propose a new parametrized family of joint possibility distributions for a given pair of fuzzy numbers. The (Pompeiu–Hausdorff) norms of the respective sup-J extensions of the sum can be adjusted by modifying the parameter. This paper proves that the norms cover the entire interval ranging from the smallest possible value to the largest one. Finally, we apply our approach to numerical methods for solving first-order fuzzy initial value problems.

In this context recall, that the widths and thus the norms of the analytical solutions, which depend on the choice of the fuzzy derivative, at t of many fuzzy differential equations such as decay processes tend to zero for $t \to \infty$ [10–13]. Note that numerical methods for solving fuzzy initial value problems based on the widely used Runge–Kutta numerical method involve additions. On the one hand, employing the usual addition based on Zadeh's extension for this purpose causes a rapid growth of the widths and the norms of the numerical solution at t for $t \to \infty$. This phenomenon may not reflect the qualitative behavior of the analytical solution. On the other hand, this effect may be mitigated by using an addition based on the sup-J extension principle with a suitable joint possibility distribution. This problem motivated us to devise the aforementioned parametrized approach towards adding fuzzy numbers that allows for controlling the Pompeiu–Hausdorff norm of the resulting sum. This way, one is able to produce a numerical solution whose norm can be adjusted as desired. Note that the width of a fuzzy number A is bounded from above by $2||A||$ and therefore one can obtain control over $width(A)$ by tuning the Pompeiu–Hausdorff norm of A.

The paper is organized as follows. Section 2 provides some mathematical background. Section 3 reviews some approaches towards adding fuzzy numbers using the sup-J extension principle. Section 4 introduces a new parametrized family of sup-J extensions of the sum whose norms are increasing with respect to the parameter γ. We present some applications to fuzzy initial value problems in Section 5 and finish with some concluding remarks in Section 6. The proofs of the theorems and lemmas are contained in the appendix.

2. Mathematical background

A fuzzy (sub)set A over a universe of discourse X is determined by its membership function $\mu_A : X \to [0, 1]$ where $\mu_A(x)$ represents the membership degree of x in A [14,15]. For notational convenience, we simply write $A(x)$ instead of $\mu_A(x)$ for all $x \in X$. The symbol $\mathcal{F}(X)$ denotes the class of fuzzy sets over the universe X. The class of fuzzy sets extend the power set of X, denoted by $\mathcal{P}(X)$, since every subset Y of X is uniquely determined by its characteristic function χ_Y.

For any pair of partially ordered sets $\mathbb{L}$ and $\mathbb{M}$, a function $f : \mathbb{L} \to \mathbb{M}$ is called increasing if $x \le y \in \mathbb{L}$ implies that $f(x) \le f(y) \in \mathbb{M}$. Similarly, f is called decreasing if $x \le y$ implies that $f(y) \le f(x)$.

Recall that $\mathcal{F}(X)$ together with the operators of infimum and supremum given by the intersection and union operators of fuzzy sets represents a complete lattice, that is, a partially ordered set that contains the infimum as well as the supremum of every subset [16]. From another point of view, one perceives that the complete lattice structure of $\mathcal{F}(X)$ is induced by the complete lattice structure of the totally ordered set $[0, 1]$ [17]. If M is a subset of an arbitrary complete lattice $\mathbb{L}$, then the symbols $\bigwedge M$ and $\bigvee M$ denote respectively the infimum and the supremum of M. One also writes $0_{\mathbb{L}}$ instead of $\bigwedge \mathbb{L}$ and $1_{\mathbb{L}}$ instead of $\bigvee \mathbb{L}$. Note that $\bigvee \emptyset = 0_{\mathbb{L}}$ and $\bigwedge \emptyset = 1_{\mathbb{L}}$ in an arbitrary complete lattice $\mathbb{L}$. In particular, we will make use of the fact that $\bigvee \emptyset = 0$ if $\mathbb{L} = [0, 1]$.

Operators of fuzzy logic arises from the generalization of operators of Boolean logic to the unit interval $[0, 1]$. Here, we will only employ the extensions of the "and" operator known as *t-norms*. Mathematically speaking, a t-norm is a commutative, associative, and increasing function $t : [0, 1]^2 \to [0, 1]$ with identity element 1. The minimum operator (that corresponds to the infimum of a two-element set) and the drastic t-norms, denoted respectively by the symbols $\wedge$ and t_D, constitute well-known examples of t-norms. Recall that the drastic t-norm is given for all $a, b \in [0, 1]$ by

$$a \, t_D \, b = \begin{cases} a \wedge b, & \text{if } a = 1 \text{ or } b = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy relation between arbitrary sets X and Y is given by a fuzzy set over the universe $X \times Y$, that is, a function $R : X \times Y \to [0, 1]$. The value $R(x, y)$ can be interpreted as the degree of relationship between x and y for $(x, y) \in X \times Y$ [18]. A fuzzy set R over a universe of the form $X = X_1 \times \ldots \times X_n$ can be interpreted as a (n-ary) fuzzy relation