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# Necessary and sufficient conditions for the equality of interactive and non-interactive extensions of continuous functions

Lucian Coroianu [a], Robert Fullér [b,c]

[a] *Department of Mathematics and Computer Science, University of Oradea, Romania*
[b] *Department of Informatics, Széchenyi István University, Győr, Hungary*
[c] *Institute of Applied Mathematics, Óbuda University, Budapest, Hungary*

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**Abstract**

In this contribution we find the class of n-dimensional joint possibility distributions with the property that the interactive extension principle coincides with the non-interactive extension principle as long as the interactive operations are determined by continuous functions strictly increasing in each argument. This result completes recent studies by the authors, where the particular case of interactive additions and multiplications versus non-interactive additions and multiplications were investigated. In addition, this time we propose results that also cover the cases when we know the fuzzy numbers only from their membership functions. It means that we eliminated the limitations that appear when we cannot pass from membership function representation to parametric representation of fuzzy numbers. As important new applications, we mention the study on the completely correlated fuzzy numbers. Also of note is that we propose two simple methods to extend bidimensional joint possibility distributions to n-dimensional joint possibility distributions. One method is based on an inductive construction while the other one is based on a pairwise construction.


*Keywords:* Fuzzy number; Joint possibility distribution; Interactive extension principle

## 1. Introduction

Obviously, when we discuss about fuzzy arithmetic, the first thing that comes in our mind is Zadeh's extension principle. On the other hand, starting with the 80's, researchers tried other ways to obtain operations with fuzzy numbers. In order to avoid repetition, we refer to paper [4], where a detailed discussion on this issue can be found in the Introduction. What is certain, is that two important trends exist in the literature. One is based on the so-called triangular norm-based extension principle and the other one is called extension principle based on joint possibility distributions. Please note that both methods are also referred as interactive extension principles, in the sense that we have an interaction between fuzzy numbers through the triangular norm or the joint possibility distribution. Both methods, actually, generalize the extension principle since by a suitable choice of the triangular norm, respectively

*E-mail addresses:* lcoroianu@uoradea.ro (L. Coroianu), fuller.robert@nik.uni-obuda.hu (R. Fullér).

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of the joint possibility distribution, we get the standard extension principle. The 80's and 90's are the years where triangular norm-based extensions were a front line topic in fuzzy arithmetic (see e.g. [7], [10], [12], [13], [15] and see also [4] for a richer reference list). But this topic is still of interest as one can find contributions in the near past (see e.g. [14], [21], [19]). In 2004, Fullér and Majlender (see [11]) introduced so called joint possibility distributions of fuzzy numbers, which can be viewed as Possibility Theory alternatives of the joint probability distributions. Joint possibility distributions have been applied in statistical type problems (see [1], [2] and other references which can be found e.g. in [4]), multi-period portfolio selection (see [23], [24]), but also in fuzzy arithmetic. More precisely, in [1] an interactive extension principle is proposed based on a joint possibility distributions. Actually, this extension agrees with the triangular norm-based extensions since the later ones can be obtained from suitably chosen joint possibility distributions. Let us recall that an important application can be found in paper [1], where the authors found a joint possibility distribution, for which one fuzzy number is actually the additive inverse of the other, where both fuzzy numbers are not crisp. It is well known that such property does not hold with respect to non-interactive addition. What is common to all papers, is that although definitions are presented in general n-dimension, almost all theoretical results and definitely all applications (at least in fuzzy arithmetic) employ only the case of bidimensional joint possibility distributions. Therefore, this paper addresses a problem in the very general setting of n-dimensional joint possibility distributions. More exactly, we will find necessary and sufficient conditions for the equality between the interactive and non-interactive outputs of the extension principle. We believe it is important to find out the cases when the two extensions coincide. This also makes possible to choose such joint possibility distributions for which the output is really interactive as it does not coincide with the output given by the non-interactive extension principle. We will consider a setting in which operations are generated by a continuous function strictly increasing in each argument. It is important to mention that similarly to the case of extensions based on triangular norms, interactive extensions based on joint possibility distributions satisfy the property that each $\alpha$-cut of the output is included in the $\alpha$-cut of the output given by the non-interactive extension principle. Therefore, the equality of the two outputs should be regarded as a limiting behavior. Now, in the case of extensions based on triangular norms, there are important studies where analytical formulae are given for the $\alpha$-cuts of the outputs (see [10], [20], [22]). In particular, from these results it is quite easy to check when the extensions given by triangular norms and those given by the standard extension principle will give the same output. Actually, from Corollary 5.1 in [22], it is quite easy to prove that the two types of extensions cannot coincide unless the triangular norm is the minimum operator. But of course, extensions generated by the minimum norm are nothing else but non-interactive extensions. Therefore, in the case of triangular norm-based extensions we have a trivial solution for this problem. When we consider extensions based on joint possibility distributions, we do have non-trivial solutions of this problem. Therefore, these types of extensions are more flexible. On the other hand, it would be really important to find analytical formulae for the $\alpha$-cuts of the outputs obtained from extensions based on joint possibility distributions completing the results obtained in the special case of triangular norm-based extensions. We will address this problem in a future contribution. Now, returning to the present contribution, the idea is to compare the outputs obtained from extensions based on joint possibility distributions with the outputs given by the non-interactive extension principle. The first paper where such problem was addressed is [1]. The authors considered the particular case of addition and the question was under which conditions the interactive and non-interactive sums of two fuzzy numbers coincide. This problem was solved in paper [4], where necessary and sufficient conditions were proposed in order to obtain this equality. Similar results were obtained in case of multiplication (see [5]). Now, obviously addition of reals or multiplication of strictly positive reals, is nothing else but a particular continuous and strictly increasing in each argument binary operation. Therefore, taking into account all the facts discussed just above, a more general approach is when we consider an arbitrary function in n variables which is continuous and strictly increasing in each argument. This is what we will do in this paper, and so the context will be very general as we consider n-dimensional joint possibility distributions and functions in n variables, with the aforementioned properties. As an interesting remark, let us mention that comparison between interactive (based on joint possibility distributions) and non-interactive operations is discussed in paper [17] too, but in that paper the discussion is with respect to so called discrete fuzzy numbers, that is, fuzzy sets having a finite set as support.

The paper is organized as follows. Section 2 presents generalities on fuzzy numbers and joint possibility distributions. Section 3 contains the main results, that is, necessary and sufficient conditions for the equality between non-interactive based operations and joint possibility distribution-based interactive operations determined by continuous and strictly increasing in each argument functions. We will study the problem locally, at some point of the domain, but also globally on the whole domain. Analyzing the main results, we arrive at the same conclusion as in the case of

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