Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Fuzzy Sets and Systems ••• (••••) •••–•••
FUZZY sets and systems
www.elsevier.com/locate/fss

A technique for fuzzifying metric spaces via metric preserving mappings

Valentín Gregori [a], Juan-José Miñana [b], Oscar Valero [b]

[a] *Instituto de Matemática Pura y Aplicada, Universitat Politècnica de València, Campus de Gandia, Calle Paranimf 1, 46730 Gandia, Spain*
[b] *Departamento de Ciencias Matemáticas e Informática, Universidad de las Islas Baleares, Carretera de Valldemossa km. 7.5, 07122 Palma, Spain*

Received 21 December 2016; received in revised form 19 July 2017; accepted 24 July 2017

Abstract

In this paper we develop a new technique for constructing fuzzy metric spaces, in the sense of George and Veeramani, from metric spaces and by means of the Lukasievicz t-norm. In particular such a technique is based on the use of metric preserving functions in the sense of J. Doboš. Besides, the new generated fuzzy metric spaces are strong and completable, and if we add an extra condition, they are principal. Appropriate examples of such fuzzy metric spaces are given in order to illustrate the exposed technique.

Keywords: Fuzzy metric space; Completion; Metric preserving function; Stationary; Strong (non-Archimedean); Principal

1. Introduction

In 1994 George and Veeramani introduced in [3] a concept of fuzzy metric space. Since then, several authors has been studied deeply this concept both from the mathematical point of view (see, for instance, [8,15,16,18]) and from the applied viewpoint (see, for instance, [1,13,14,12]).

Regarding the mathematical point of view, in [8], V. Gregori and S. Romaguera showed that metrizable topological spaces coincide with fuzzy metrizable topological spaces. It follows that topologically, metric and fuzzy metric spaces are identical. However, we can find differences between these two concepts when we focus our attention on the intrinsic (fuzzy) metric properties. An instance of such differences that are worth mentioning is provided in [9] where Gregori and Romaguera proved that there exist fuzzy metric spaces which do not admit completion. An important class of fuzzy metric spaces that are not in general completable are the so-called fuzzy stationary metric spaces (see Definition 2.9). This type of fuzzy metric spaces are the closest to the classical metrics and, in fact, many properties can

E-mail addresses: vgregori@mat.upv.es (V. Gregori), jj.minana@uib.es (J.-J. Miñana), o.valero@uib.es (O. Valero).

http://dx.doi.org/10.1016/j.fss.2017.07.022

Please cite this article in press as: V. Gregori et al., A technique for fuzzifying metric spaces via metric preserving mappings, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.07.022

be extended from metrics to stationary fuzzy metrics in a straightforward manner, specially when the stationary fuzzy metric is defined by means of the Lukasievicz t-norm $\mathfrak{L}$. It is due to the fact that stationary fuzzy metrics enjoy two distinguished properties, they are principal (see Definition 2.12) and strong (non-Archimedean) (see Definition 2.18).

In the matter of applications, a few techniques used in image filtering and in the study of perceptual color difference have been improved when a classical metric has been replaced by a fuzzy metric. Nonetheless, it must be pointed out that the shortage of examples of fuzzy metrics in the literature turns be a drawback when one wants to apply fuzzy metrics to the aforesaid engineering problems.

The aim of this paper is twofold: On the one hand, inspired by the fact that there are fuzzy metric spaces that are not completable, we develop a technique for constructing fuzzy metric spaces from metric spaces and by means of the Lukasievicz t-norm which are completable. In particular such a technique is based on the use of metric preserving functions in the sense of J. Doboš ([2]). Besides, the new generated fuzzy metric spaces are strong and when we add an extra condition they are also principal. Furthermore, we show that some well-known examples can be obtained using our technique. On the other hand, motivated for the aforementioned lack of examples, new examples can be constructed applying our new technique in order to overcome the mentioned drawback.

The paper is organized as follows. Section 2 is devoted to recall the basic notions that will be crucial throughout the paper. In Section 3, we introduce the notion of uniformly continuous mapping between stationary fuzzy metric spaces and metric spaces, and *vice-versa*. Thus we define when they are equivalent. Based on such a notion, we present a technique that allows to construct stationary fuzzy metric spaces from a metric space by means of metric preserving functions with values in $[0, 1[$. Moreover, it is showed that the new constructed stationary fuzzy metric spaces are completable provided that the used metric preserving function is a strongly metric preserving function (in the sense of Doboš). In addition, it is proved that the new stationary fuzzy metric spaces are complete if and only if the metric spaces from which are generated are also complete. Section 4 is devoted to generalize the construction presented in Section 3 to the non-stationary case. Thus fuzzy metric spaces are generated from metric spaces by means of a family of metric preserving functions that satisfy a distinguished condition which will be specified later on. These fuzzy metric spaces are always strong and, in addition, they are complete if and only if the metric spaces from which are generated are also complete. Furthermore, they are principal and completable whenever the set of all metric preserving functions belonging to the family under consideration are strongly metric preserving functions. Appropriate examples that illustrate the exposed theory are also yielded.

2. Preliminaries

In the following we will recall the notions that will be crucial in our subsequent work. With this aim, we will divide this section in two parts. In the first part, we will recall those notions related to metric preserving functions. In the second one, we will fix the pertinent notions about fuzzy metric spaces in which our work will be based on.

2.1. Metric preserving functions

We recall, according to Doboš, the basic and pertinent notions about metric preserving functions (for a detailed treatment we refer the reader to [2]).

Let (X, d) be a metric space. For each $f : [0, \infty[\to [0, \infty[$ denote by d_f the function $d_f : X \times X \to [0, \infty[$ defined as follows

$$d_f(x, y) = f(d(x, y)) \text{ for each } x, y \in X.$$

In the light of the preceding construction we are able to introduce the notion of metric preserving function.

Definition 2.1. A function $f : [0, \infty[\to [0, \infty[$ is a *metric preserving* if for each metric space (X, d) the function d_f is a metric on X.

From now on, we will denote by $\mathcal{M}$ the class of all metric preserving functions. Moreover, we will denote by $\mathcal{O}$ the set of all functions $f : [0, \infty[\to [0, \infty[$ with $f^{-1}(0) = \{0\}$. It is obvious that $\mathcal{M} \subset \mathcal{O}$.

The next result provides a distinguished class of metric preserving functions. In order to state such a result we recall the notion of subadditive function. A function $f : [0, \infty[\to [0, \infty[$ is subadditive provided

Please cite this article in press as: V. Gregori et al., A technique for fuzzifying metric spaces via metric preserving mappings, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.07.022