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# Fuzzy quasi-pseudometrics on algebraic structures

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**Abstract**

In this work we state a number of theorems about fuzzy (quasi-)pseudometrizable algebraic structures. Our most useful results are: (1) a fuzzy semitopological group whose topology is induced by a left-invariant fuzzy (quasi-)pseudometric, then it is a fuzzy (paratopological) topological group, (2) if the topology on a semigroup $S$ is induced by an invariant fuzzy quasi-pseudometric, then $S$ is a fuzzy topological semigroup, and (3) the same conclusion is valid for a left-invariant fuzzy quasi-pseudometric on a monoid $G$ such that the left translations are open and the right translations are continuous at the identity $e$ of $G$. By means of the standard fuzzy (quasi-)pseudometric $M_d$ associated to a (quasi-)pseudometric $d$, our results apply in the case of semitopological groups, semigroups and monoids in order to obtain new results that allow us to generalize and to strengthen previous outcomes.


*Keywords:* Fuzzy (quasi-)pseudometric; Left topological group; Right topological group; Semitopological group; Paratopological group; Topological group; Topological semigroup; Invariant fuzzy (quasi-)pseudometric

## 1. Introduction

In this paper we shall focus our attention on topological algebraic structures equipped with a fuzzy quasi-pseudometric in the sense of Kramosil and Michalek. Combinations of a fuzzy metric structure and an algebraic structure deserve special attention in fuzzy Topological Algebra. The most frequently studied structures fall into the so-called fuzzy normed spaces (among others, the interested reader can consult [2,4,15,16]), although fuzzy metric topological groups are also worthy of consideration (see [11,19,20]). In [8] Gregori and Romaguera remove the *symmetric* condition in the definition of a fuzzy metric (in the sense of Kramosil and Michalek, see [14]) and introduce the notion of a *fuzzy quasi-metric space*. This allows us to consider nonsymmetric structures which fit in the realm of fuzzy nonsymmetric topology: fuzzy quasi-metric spaces and fuzzy quasi-normed spaces ([1,5,7,10]). In this context,

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the *classical results* in semigroups and paratopological groups related to (left-)invariant metrics (see, for example, [17, Theorem 2.1], [18, Proposition 3.2]) encourage enough to merit further investigation in fuzzy nonsymmetric topology.

In this paper we deal with one of the oldest problems in Topological Algebra: find sufficient conditions in order that a *topological algebraic* structure (in particular a *nonsymmetric* structure) become a *stronger topological structure* (in particular, a *symmetric* structure). Our goal is to obtain conditions on fuzzy quasi-metrics (respectively, fuzzy quasi-pseudometrics) semitopological groups and fuzzy paratopological groups (respectively, on fuzzy topological semigroups) which imply that they are either fuzzy paratopological groups or topological groups.

To be precise, we show the following: (1) if $(G, M, *)$ is a fuzzy semitopological group whose topology $\tau_M$ is induced by a left-invariant fuzzy (quasi-)pseudometric $M$, then $(G, M, *)$ is a fuzzy (paratopological) topological group, (2) if $(M, *)$ is an invariant fuzzy quasi-pseudometric on a semigroup $S$, then $(S, M, *)$ is a fuzzy topological semigroup, and (3) the same conclusion is valid when $(M, *)$ is a left-invariant fuzzy quasi-pseudometric on a monoid $G$ such that the left translations are open and the right translations are continuous at the identity $e$ of $G$. It is worth mentioning that, by means of the standard fuzzy (quasi-)pseudometric $M_d$ associated to a (quasi-)pseudometric $d$, our results apply in the case of semitopological groups, semigroups and monoids in order to obtain new results that allow us to generalize and to strengthen previous outcomes by Liu [17, Theorem 2.1] and Ravsky [18, Proposition 3.2].

A celebrated theorem by Birkhoff–Kakutani says that a Hausdorff topological group is metrizable if and only if it is first-countable. It is also a well-known result that a first-countable paratopological group is quasi-metrizable (see [18]). In spite of these results, a first-countable topological semigroup need not be quasi-pseudometrizable (see [12]).

## 2. Preliminaries

According to [21], a *continuous t-norm* is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (i) $*$ is associative and commutative, (ii) $*$ is continuous, (iii) $a * 1 = a$ for every $a \in [0, 1]$, and (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

It is a well-known fact, and easy to check, that for each continuous t-norm $*$ one has $* \leq \wedge$, where $\wedge$ is the continuous t-norm given by $a \wedge b = \min\{a, b\}$. The interested reader is referred to [13] for further information on (continuous) t-norms.

Following [8], a *fuzzy quasi-pseudometric* (in the sense of Kramosil and Michalek) on a set $X$ is a pair $(M, *)$ such that $M$ is a fuzzy set in $X \times X \times [0, \infty)$ and $*$ is a continuous t-norm satisfying for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, 0) = 0$;
(ii) $M(x, x, t) = 1$;
(iii) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(iv) $M(x, y, \_) : [0, +\infty) \to [0, 1]$ is left continuous.

A *fuzzy pseudometric* on $X$ is a fuzzy quasi-pseudometric $(M, *)$ on $X$ which satisfies:

(v) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$.

By a *fuzzy (quasi-)pseudometric space* (in the sense of Kramosil and Michalek) we mean a triple $(X, M, *)$ such that $X$ is a set and $(M, *)$ is a fuzzy (quasi-)pseudometric on $X$. Every fuzzy (quasi)-pseudometric $(M, *)$ on a set $X$ induces a topology $\tau_M$ on $X$ having as a base the family $\{B_M(x, \varepsilon, t) : x \in X,\ \varepsilon \in (0, 1),\ t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$.

Let $(X, d)$ be a (quasi-)pseudometric space. Define a fuzzy set $M_d$ in $X \times X \times [0, \infty)$ by

$$M_d(x, y, t) = \begin{cases} \dfrac{t}{t + d(x, y)} & \text{for all } x, y \in X \text{ and } t > 0; \\ 0 & \text{for all } x, y \in X \text{ and } t = 0. \end{cases}$$

Then $(M_d, \wedge)$ is a fuzzy (quasi-)pseudometric on $X$, and hence $(M_d, *)$ is a fuzzy quasi-pseudometric on $X$ for all continuous t-norm $*$, the so-called (fixed a t-norm $*$ on $X$), the standard fuzzy (quasi-)pseudometric induced by $d$ on $X$. It is known that the topology $\tau_M$ and the topology $\tau_d$, induced by the (quasi-)pseudometric $d$, coincide (see [6,8,9]).

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