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Stratified LMN-convergence tower groups and their stratified LMN-uniform convergence tower structures

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Received 31 March 2016; received in revised form 18 January 2017; accepted 19 January 2017

This article is dedicated to Professor Robert Lowen on his 70th birthday

Abstract

If L and M are frames, and N is a quantale, then using stratification mappings between frames, we introduce a category of stratified LMN-convergence tower groups – a topological category. We then prove that every stratified LMN-limit tower group induces a stratified LMN-uniform convergence tower space. Also, we introduce a category of stratified LMN-Cauchy tower groups, and show that the category of strongly normal stratified LMN-limit tower groups, is isomorphic to the category of stratified LMN-Cauchy tower groups. We provide various examples in support of our theories so far developed herein the text.

Keywords: Quantale; Topology; Convergence space; Convergence group; Convergence tower group; Cauchy convergence group; Uniform convergence space; Category theory

1. Introduction

Starting with a fixed basis lattice, a frame and later, with an enriched cl-premonoid [22], we studied various convergence structures on groups, obtaining several interesting results on the category of stratified lattice-valued convergence groups [3,4] and probabilistic convergence groups [7,32]. In doing so, we used the notion of a stratified framed-valued fuzzy convergence structure studied by G. Jäger [23,40] that arose from the notion of stratified lattice-valued filters defined by Höhle and Šostak [22]; also, using probabilistic convergence structure that arises from [27,45,49].

Very recently, G. Jäger [28,29] (see also [52]) proposed a *stratification mapping between frames* to help build so-called s-stratified LMN-convergence tower structures, L, M being frames, and N being a complete lattice or a quantale; this *stratification mapping* along with some additional conditions plays a crucial role in many instances. One of the striking aspects of this theory of s-stratified LMN-convergence tower spaces is that it captures various con-

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http://dx.doi.org/10.1016/j.fss.2017.01.011

vergence structures such as probabilistic convergence spaces [6,27,45,48], convergence spaces [11,43], lattice-valued convergence spaces [12,14,15,21,23,24,37,38,41,42,50,51].

The motive behind the present article is to explore the use of stratification mappings between different frames in an attempt to study various s-stratified LMN-convergence structures on groups and look at the categories of s-stratified LMN-convergence tower groups, s-stratified LMN-limit tower groups, s-stratified Cauchy tower groups in a more general fashion than the attempt made previously on various convergence structures in conjunctions with groups. This, however, paves the way to *overlooking* the previous work on lattice-valued convergence groups and beyond from a common view point.

We organized our work as follows: in a preliminary section we put some basic facts of lattice theory including the notion of *stratification mapping* with examples that dominate the article; and in Sections 3 and 4, we continue to collect the notions of LM-filters along with some related results, and also, provide some essential results on s-stratified LMN-convergence structures introduced and studied in [28,29]. In Section 5, we study s-stratified LMN-Cauchy tower spaces. In Section 6, we introduce a notion of s-stratified LMN-convergence tower groups, some related properties and characterization, while Section 7 deals with s-stratified LMN-uniformizability of s-stratified LMN-limit tower groups, and in Section 8, we introduce a notion of s-stratified LMN-Cauchy tower group and study the link between this notion and the notion of s-stratified LMN-limit tower group. In each section, we provide a good number of stimulating examples in support of our endeavor.

2. Preliminaries

Let $L = (L, \leq, \wedge)$ be a complete lattice, [19,33]. If, moreover, a complete lattice N is equipped with a semigroup operation $*\colon N \times N \longrightarrow N$ satisfying $\alpha * (\bigvee_{i\in J} \beta_i) = \bigvee_{i\in J}(\alpha * \beta_i)$ and $(\bigvee_{i\in J} \beta_i) * \alpha = \bigvee_{i\in J}(\beta_i * \alpha)$, then $(N, \leq, *)$ is called a *quantale*. A quantale $(N, \leq, *)$ is called *commutative* if the underlying semigroup $(N, *)$ is commutative; furthermore, if it satisfies $\top^N * \alpha = \alpha * \top^N = \alpha$, $\forall \alpha \in N$, then it is called *integral* [20,22,46]. Unless otherwise mentioned, we consider herein this text that $(N, \leq, *)$ is a commutative integral quantale. We denote the category of commutative integral quantales and quantale-homomorphisms by **Quant**. Typical examples of quantales are the following.

- For a left-continuous t-norm $*$ on $[0, 1]$ (see [36]), $([0, 1], \leq, *)$ is a quantale.
- Let Δ^+ be the set of *distance distribution functions*, see [47]. If we consider a sup-continuous triangle function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, then $(\Delta^+, \leq, \tau)$ is a quantale.
- Consider $[0, \infty]$ with the opposite order and the usual addition, extended by $x + \infty = \infty + x = \infty$, then $([0, \infty], \geq, +)$ is a quantale.
- A *frame* $(L, \leq, \wedge)$ is a quantale with $* = \wedge$. It is commutative, idempotent, and integral. Let **Frm** denote the category of frames and frame-homomorphisms. We note that for any quantale we have $\alpha * \beta \leq \alpha \wedge \beta$ and that if the quantale operation is idempotent, i.e., $\alpha * \alpha = \alpha$ for all $\alpha \in L$, then $* = \wedge$.

For a commutative quantale and, so for a frame, the implication operation $\rightarrow\colon L \times L \longrightarrow L$, called *residuum* is given by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L\colon \alpha * \gamma \leq \beta\}$. For a detailed account on this operation we refer to [20,22,36].

For a frame L with top element $\top^L$ and bottom element $\perp^L$ and a set X, we denote the set of all L-sets on X by $L^X (= \{\nu\colon X \longrightarrow L\})$. If $A \subseteq X$, then a constant L-set with value $\alpha \in L$, is denoted by α_A, and is defined as $\alpha_A(x) = \alpha$, if $x \in A$ and $\alpha_A(x) = \perp^L$, elsewhere. In particular, we write $\top^L_X$, resp. $\perp^L_X$, for the constant L-sets with value $\top^L$, resp. $\perp^L$. The lattice operations are extended pointwise from L to L^X, i.e., we define $(\nu \wedge \mu)(x) = \nu(x) \wedge \mu(x)$, $(\nu \vee \mu)(x) = \nu(x) \vee \mu(x)$, $\left(\bigwedge_{i\in J} \nu_i\right)(x) = \bigwedge_{i\in J} (\nu_i(x))$, $\left(\bigvee_{i\in J} \nu_i\right)(x) = \bigvee_{i\in J}(\nu_i(x))$. For a mapping $f\colon X \longrightarrow Y$, $\nu \in L^X$ and $\mu \in L^Y$, the image of ν under f, $f^{\rightarrow}(\nu) \in L^Y$, is defined by $f^{\rightarrow}(\nu)(y) = \bigvee_{f(x)=y} \nu(x)$, for $y \in Y$, and the pre-image of μ under f, $f^{\leftarrow}(\mu) \in L^X$, is defined by $f^{\leftarrow}(\mu) = \mu \circ f$. If $\nu \in L^X$ and $s\colon L \longrightarrow M$ is a mapping between the frames L and M, we define $s(\nu) \in M^X$ by $s(\nu)(x) = s(\nu(x))$, $x \in X$.

If $L, M \in |\mathbf{Frm}|$, then a mapping $s\colon L \longrightarrow M$ is called a *stratification mapping* if it satisfies (SM1) $s(\perp^L) = \perp^M$, (SM2) $s(\top^L) = \top^M$ and (SM3) $s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$ for all $\alpha, \beta \in L$, where $\perp^L$ is the bottom element in L, similarly $\perp^M$ is the bottom element in M; while $\top^L$ denotes top element in L and $\top^M$ is the top element in M. Note that due to (SM3), the stratification mapping s is non-decreasing.

Please cite this article in press as: T.M.G. Ahsanullah, G. Jäger, Stratified LMN-convergence tower groups and their stratified LMN-uniform convergence tower structures, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.01.011