Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 276 (2015) 59–73
FUZZY sets and systems
www.elsevier.com/locate/fss

The existence of vague objects

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Received 25 October 2014; received in revised form 25 April 2015; accepted 27 April 2015

Available online 4 May 2015

Abstract

We consider a category whose objects are defined as triplets $(S, e, \underline{c})$ where S is a set, e is a similarity and $\underline{c} \in S$. An object is called *crisp* if the similarity is crisp, *vague* otherwise. We indicate different ways of obtaining a similarity and therefore different ways to obtain an object. The paper is mathematical in nature, nevertheless I hope that the proposed formalisms are in some ways connected with the very interesting ontological question about the existence of vague objects in real world.

Keywords: Vague objects; Category theory; Ontology; Similarity; Fuzzy subgroups; Fuzzy logic; Ontic vagueness

1. Introduction

Does vagueness exist *in sola lingua*, or also *in re*? In particular, is it reasonable to admit that there are *vague objects* in the world? These questions are very basic both for ontology (see [3–5]) and for understanding the phenomenon of vagueness (see [1,2,6,7,11,16–22,24]). They originate a significant debate in the community of philosophers. Usually, starting from Frege and Russell [24], peoples are rather unwilling to accept the existence of vagueness in real world. For example A. Varzi [28], in putting the questions "*What exactly is Everest? Where does it begin and where does it end?*", accepts that the phenomenon of the vagueness occurs in connection with mount Everest. Nevertheless it rejects the hypothesis for which the term 'Everest' denotes a vague object. This since:

> "[vagueness] *lies in the representation system* (*our language, our conceptual apparatus*), *not in the represented entity*, … *one mistakenly infers that the product of a representation is vague because the representation process is vague* …"

and again

> "*to say that the referent of a geographic term is not sharply demarcated is to say that the term vaguely designates an object, not that it designates a vague object*".

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http://dx.doi.org/10.1016/j.fss.2015.04.015
0165-0114/

Obviously, there are also philosophers with a totally different opinion, for example Tye [25] says "*I believe that there are vague objects...*".

While this paper was inspired from these intriguing questions, it is mathematical in nature since confines itself to propose a mathematical model to represent the idea of vague object. Namely, a category is defined whose objects (in the sense of category theory) are defined by an element $\underline{c}$, the *prototype*, a set S containing $\underline{c}$, *the set of possible variation*, and a similarity (the notions of prototype and similarity are considered for example by Rosch in [23] and by Vetterlein in [31]). An object is crisp if the similarity is crisp, vague otherwise (see also [13]). This category is related with well known categories proposed in literature to give a solid foundation for fuzzy set theory (see for example [17] and [18]).

2. Abstract entities in mathematics

I start by emphasizing a general schema used in mathematics to define new abstract entities from pre-existing ones. This schema is based on the *abstraction process* consisting in the passage from a structure S to a quotient $S/\equiv$ of S modulo a congruence $\equiv$. An object in this new structure is an equivalence class. Perhaps a similar process is also on the basis of the first steps in defining the mathematical structures from the experience. Now, from a technical point of view there is no difference in considering instead of the structure $(S/\equiv, =)$ where $=$ is the identity, the structure $(S, \equiv)$ where $\equiv$ is a congruence. In this case we have to consider as an object a pair $(x, \equiv)$ where $x \in S$. In other words, in several cases

we can view an object in mathematics as at an element in a set together with an equivalence criterion.

In the next sub-sections I list some examples.

2.1. Rational numbers

The field of rational numbers is defined from the ring $(Z, +, \cdot)$ of integers as follows:

1. one considers the set $Fr = Z \times (Z - \{0\})$ whose elements we call *fractions*, and one defines two operations $+$ and $\cdot$ in Fr by setting

$$(n,m) + (p,q) = (n \cdot q + m \cdot p, mq); \qquad (n,m) \cdot (p,q) = (n \cdot p, m \cdot q)$$

2. one defines a congruence $\equiv$ in $(Fr, +, \cdot)$ by setting

$$(n,m) \equiv (n', m') \Leftrightarrow n \cdot m' = m \cdot n'$$

3. one considers the *quotient* $(Q, +, \cdot)$ of $(Fr, +, \cdot)$ *modulo* $\equiv$.

In brief, the field $(Q, +, \cdot)$ of rational numbers is defined by the equations

$$\begin{aligned}
&[(n,m)] = \{(n', m') \in Z \times (Z - \{0\} : (n', m') \equiv (n,m)\} \quad \text{for every } (n,m) \in Fr \\
&Q = \{[(n,m)] : (n,m) \in Z \times (Z - \{0\})\} \\
&[(n,m)] + [(p,q)] = [(n \cdot q + m \cdot p, mq)]; \\
&[(n,m)] \cdot [(p,q)] = [(n \cdot p, m \cdot q)].
\end{aligned}$$

A way to reformulate such a construction is to skip out step 3 and to interpret the identity symbol "=" by the relation $\equiv$. This means to consider the structure $(Z \times (Z - \{0\}), +, \cdot, \equiv)$ and therefore to identify a rational number with a fraction $\underline{c} = (n, m)$ together with the equivalence $\equiv$ defined in step 2. Notice that *Mathematica* embraces this point of view but it implements $\equiv$ by a system of rewriting rules and by a consequent normal form reduction.

In an analogous way we can define the ring Z of integer numbers by starting from the natural numbers.