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The bounded convergence in measure theorem for nonlinear integral functionals ☆

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Abstract

In this paper, we introduce a new notion of the perturbation of nonlinear functionals to formulate a functional form of the convergence theorems for nonlinear integrals in nonadditive measure theory. As its direct consequences, we obtain the bounded convergence in measure theorems for typical nonlinear integrals, which show that the autocontinuity of a nonadditive measure is equivalent to the validity of the bounded convergence in measure theorems for the Choquet, the Sugeno, and the Shilkret integrals as well as their symmetric and asymmetric extensions.

Keywords: Nonadditive measure; Choquet integral; Sugeno integral; Shilkret integral; Perturbation; Bounded convergence theorem

1. Introduction

In nonadditive measure theory, several types of nonlinear integrals have been proposed. Among them, the Choquet integral [1], the Sugeno integral [13], and the Shilkret [12] integral are typical and widely used in theory as well as its practical applications [2,5,8,17].

The convergence in measure theorem for the integral, that is, the convergence theorem for the integrals of a sequence of measurable functions converging in measure, is one of interesting research objects in nonadditive measure theory since the validity of such convergence theorems is closely related to the structural characteristics for nonadditive measures, that is, the autocontinuity. This notable relationship was revealed by a theorem of Wang [14, Theorem 16], which shows that, if a nonadditive measure is continuous, that is, a fuzzy measure, then its autocontinuity is equivalent to the validity of the convergence in measure theorem for the Sugeno integral; see also [15, Theorem 5.3]. Its extension to the generalized fuzzy integral, which contains both of the Sugeno integral and the Shilkret integral, was given in Xie and Fang [19, Theorem 1 and Corollaries 1 and 2]. The convergence in measure theorem for the Choquet integral was also given in Wang [16, Theorem 7] for an equi-integrable sequence of functions.

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In Murofushi et al. [7, Theorem 3.3], they focused on the convergence in measure theorem for a uniformly essentially bounded sequence of functions, which we shall call the bounded convergence in measure theorem (for short, BCM theorem), and proved that the autocontinuity of a nonadditive measure is equivalent to the validity of the BCM theorem for (the asymmetric extension of) the Choquet integral without assuming the continuity of the measure.

The purpose of the paper is to formulate the BCM theorem for a general nonlinear functional, which yields the BCM theorem for nonlinear integrals such as the Choquet integral, the Sugeno integral, the Shilkret integral, and their symmetric and asymmetric extensions.

The Lebesgue integral may be viewed as the linear functional, determined by the Lebesgue measure, on the space of measurable functions. By contrast, the Choquet, the Sugeno, and the Shilkret integrals are nonlinear functionals since they are determined by nonadditive measures. Therefore, in order to discuss the BCM theorem regardless of the type of such nonlinear integrals and develop a unifying approach to the proof, it is reasonable to formulate it for a general nonlinear functional. In this formulation, a key concept is the perturbation of a nonlinear functional, which manages the small change of the value of the functional by adding a small term to a measurable function in the domain of the functional as well as to a measure that determines the functional (Definition 3.3).

The paper is organized as follows. In Section 2, we recall some definitions on nonadditive measures and essential boundedness of functions. We also collect some definitions and basic properties of the Choquet, the Sugeno, the Shilkret integrals, and their symmetric and asymmetric extensions. In Section 3, we introduce the perturbation property of a nonlinear functional and show that the Choquet, the Sugeno, the Shilkret integrals and their asymmetric extensions enjoy this property. We also give an example showing that this key property is not satisfied for the symmetric extensions. In Section 4, we formulate the BCM theorem for a nonlinear functional having the perturbation property, which yields the BCM theorems for the Choquet, the Sugeno, and the Shilkret integrals as well as their symmetric and asymmetric extensions. In Section 5, we present conclusions.

2. Preliminaries

2.1. *Nonadditive measure and essential boundedness of functions*

In what follows, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers, respectively. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. As usual, we assume the standard convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset = \infty$.

We say that a set function $\mu: \mathcal{A} \to [0, \infty]$ is a *nonadditive measure* if $\mu(\emptyset) = 0$ and $\mu(A) \le \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$ and it is *finite* if $\mu(X) < \infty$. The nonadditive measure is also called a monotone measure [17], a capacity [1], or a fuzzy measure [13] in the literature. Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu: \mathcal{A} \to [0, \infty]$ and let $\mathcal{M}_b(X) := \{\mu \in \mathcal{M}(X): \mu(X) < \infty\}$. For any $\mu \in \mathcal{M}_b(X)$, its *conjugate* $\bar{\mu} \in \mathcal{M}_b(X)$ is defined by $\bar{\mu}(A) := \mu(X) - \mu(A^c)$ for every $A \in \mathcal{A}$, where A^c denotes the complement of A. See [2,8,17] for further information on nonadditive measures.

A function $f : X \to \mathbb{R}$ is said to be *$\mathcal{A}$-measurable* if $\{f \ge t\} := \{x \in X: f(x) \ge t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X: f(x) > t\} \in \mathcal{A}$ for every $t \in \mathbb{R}$. If f is $\mathcal{A}$-measurable, then so are $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. Let $\mathcal{F}(X)$ denote the set of all $\mathcal{A}$-measurable functions $f : X \to \mathbb{R}$ and let $\mathcal{F}^+(X) := \{f \in \mathcal{F}(X): f \ge 0\}$. Let $\mathcal{F}_b(X) := \{f \in \mathcal{F}(X): f \text{ is bounded}\}$ with norm $\|f\| := \sup_{x\in X} |f(x)|$ and let $\mathcal{F}_b^+(X) := \{f \in \mathcal{F}_b(X): f \ge 0\}$. Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

The following notions of essential boundedness of functions are already discussed in [2, p. 105] and [7, Definition 3.2].

Definition 2.1. Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$ and $f \in \mathcal{F}(X)$.

(1) f is said to be *μ-essentially bounded* if there is a real number $r > 0$ such that $\mu(\{f \ge r\}) = 0$ and $\mu(\{f \ge -r\}) = \mu(X)$ and $\mathcal{F}$ is said to be *uniformly μ-essentially bounded* if there is a real number $r > 0$ such that $\mu(\{f \ge r\}) = 0$ and $\mu(\{f \ge -r\}) = \mu(X)$ for all $f \in \mathcal{F}$.