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On the composition of fuzzy power relations ☆

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Abstract

This paper deals with the "powering" of binary fuzzy relations (i.e. lifting a fuzzy relation R defined on a set X to the relation R^+ defined on the set $\mathcal{F}(X)$ of all fuzzy subsets of X). We prove that for any complete residuated lattice $\mathcal{L}$, the composition of the powers of two $\mathcal{L}$-relations is always a subset of the power of their composition. Answering to a question posed by Georgescu, we prove that the converse is not always true. We prove that the composition of the powers of two $\mathcal{L}$-relations is equal to the power of their composition if and only if $\mathcal{L}$ is a Heyting algebra.

Keywords: Power structure; Fuzzy relation; Composition of relations; Residuated lattice

1. Introduction

It is well known that several operations associated to a binary $\mathcal{L}$-relation $R : X \times Y \to L$ between sets in the framework of many-valued logic are largely used in formal concept analysis and in theoretical computer science. Their basic aspects and properties have been studied by many authors with different notation and terminology ([1,2, 6–10]). These operators are closely related to various compositions of binary $\mathcal{L}$-relations which, in their turn, depend on the basic operations of the structure of truth values $\mathcal{L}$.

In [9] the so called *fuzzy power algebras* are studied in the framework of fuzzy set theory based on a continuous t-norm. Among other things, for any binary $\mathcal{L}$-relation $R : X \times X \to L$, fuzzy relations $R^{\rightarrow}$, $R^{\leftarrow}$ and R^+ on L^X (on the set of all $\mathcal{L}$-subsets of X) are defined and the properties of these "power constructions" are studied. The notions of good, very good, Hoare good and Smyth good fuzzy relations are defined and some connections between them are established, generalizing some results from [3,4] and [5]. In [14] these fuzzy power constructions are introduced and studied in the more general case, when $\mathcal{L}$ is a complete residuated lattice.

The main aim of the present paper is to give an answer to an open question posed in [9]. Namely, in the crisp case, it is proved in [5] that for any two binary relations R and Q, $R^+ \circ Q^+ = (R \circ Q)^+$. In [9] it is announced that, if $\mathcal{L}$ is the real unit interval [0, 1] endowed with a continuous t-norm, the inequality $R^+ \circ Q^+ \leq (R \circ Q)^+$ could be proved for

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all $\mathcal{L}$-relations R and Q. Also, it has been asked whether the converse is true i.e. is it true that for any two $\mathcal{L}$-relations R and Q, the inequality $(R \circ Q)^+ \leq R^+ \circ Q^+$ holds? In this paper we prove that the inequality $R^+ \circ Q^+ \leq (R \circ Q)^+$ holds for any complete residuated lattice $\mathcal{L}$, but the converse, $(R \circ Q)^+ \leq R^+ \circ Q^+$, is not always true. We prove that, the equality $R^+ \circ Q^+ = (R \circ Q)^+$ holds for any two $\mathcal{L}$-relations R and Q if and only if $\mathcal{L}$ is a Heyting algebra.

The structure of the paper is the following: Section 2 recalls some basic definitions and properties of complete residuated lattices, fuzzy sets and the definition of crisp and fuzzy power relations $R^{\rightarrow}$, $R^{\leftarrow}$ and R^+. In Section 3 we answer the question from [9] and prove the main results.

2. Preliminaries

An $\mathcal{L}$-set (or a *fuzzy set*) in a universe X is a mapping $A : X \to L$, where L is the support of an appropriate structure $\mathcal{L}$ of truth values. Here, for $\mathcal{L}$ we choose complete residuated lattices (also known as complete commutative integral residuated lattices). Complete residuated lattices as a structure of truth values were introduced into the context of fuzzy sets and fuzzy logic by Goguen [11]. A thorough information about the role of residuated lattices in fuzzy logic can be found in [12,13,15]. If the structure $\mathcal{L}$ is fixed, then the set L^X of all $\mathcal{L}$-sets on X we denote by $\mathcal{F}(X)$, following the notation from [9] and [14].

A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\to$ satisfy the adjointness property, i.e. for all $x, y, z \in L$, $x \leq y \to z$ if and only if $x \otimes y \leq z$. If the lattice $\langle L, \wedge, \vee, 0, 1\rangle$ is complete, then $\mathcal{L}$ is a complete residuated lattice. The most applied complete residuated lattices are those with $L = [0, 1]$ (the real unit interval) with $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ and the following pairs of adjoint operations: in the *standard Lukasiewicz structure*: $a \otimes b = \max(a + b - 1, 0)$ and $a \to b = \min(1 - a + b, 1)$; in the *standard Gödel structure*: $a \otimes b = \min(a, b)$ and $a \to b = 1$ if $a \leq b$, and $a \to b = b$ otherwise; in the *standard product structure*: $a \otimes b = a \cdot b$ and $a \to b = 1$ if $a \leq b$, and $a \to b = \frac{b}{a}$ otherwise. A (complete) residuated lattice $\mathcal{L}$ is called *(complete) Heyting algebra* if for all $x, y \in L$, $x \wedge y = x \otimes y$. Note, that a residuated lattice $\mathcal{L}$ is a Heyting algebra if and only if the operation $\otimes$ is idempotent. Here are some of the properties of complete residuated lattices: for all $x, y, z, a, b \in L$

- (L1) $x \leq y$ if and only if $x \to y = 1$;
- (L2) $x \to x = x \to 1 = 0 \to x = 1$ and $1 \to x = x$;
- (L3) $x \otimes 0 = 0$;
- (L4) $x \otimes y \leq x \wedge y \leq x$;
- (L5) $x \leq y \to x$;
- (L6) $x \otimes (x \to y) \leq y$;
- (L7) $x \to y = \max\{z : x \otimes z \leq y\}$;
- (L8) if $a \leq b$ then $a \otimes x \leq b \otimes x$;
- (L9) if $a \leq b$ then $x \to a \leq x \to b$;
- (L10) if $a \leq b$ then $b \to x \leq a \to x$.

If the residuated lattice is complete, then for all $x \in L$ and any $\{y_i : i \in I\} \subseteq L$

- (L11) $x \otimes \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \otimes y_i)$;

From now on, we assume that $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice. The set L^X may be given a structure of complete residuated lattice by means of the operations $\wedge, \vee, \otimes, \to, 0_X, 1_X$ induced pointwisely by those of $\mathcal{L}$. If X is a nonempty set, a *(binary) fuzzy relation* (or an $\mathcal{L}$-relation) on X is any mapping $R : X^2 \to L$. The *transpose* (or *reverse*) of a fuzzy relation R is $R^t(x, y) = R(y, x)$, for all $x, y \in X$. If R and Q are fuzzy relations on X, their *composition* is defined by

$$(R \circ Q)(x, y) = \bigvee_{z\in X} \big(R(x, z) \otimes Q(z, y)\big).$$