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Markov kernels and tribes

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ABSTRACT

We define an ordering on the set of bounded Markov kernels associated with a tribe of fuzzy sets. We show that under this order, the set of bounded Markov kernels is a Dedekind σ-complete lattice. In addition, we define a sum of bounded Markov kernels such that the set of bounded Markov kernels is a lattice-ordered semigroup. If we concentrate only to sharp bounded Markov kernels, then this set is even a Dedekind σ-complete ℓ-group with strong unit. We show that our methods work also for bounded Markov kernels associated with T_s-tribes of fuzzy sets, where T_s is any Frank t-norm and $s \in (0, \infty)$.

1. Introduction

Markov kernels are an important tool in the classical probability theory, e.g. they are used to describe Markov processes. They have also many applications in different areas of mathematics. In [2], the authors used Markov kernels for representations of measures defined on T_s-tribes of fuzzy sets, where T_s is a Frank t-norm for $s \in [0, \infty]$. The class of Frank t-norms $\{T_s\colon s \in [0, \infty]\}$ was introduced and studied in [8] under the name fundamental t-norms. Tribes, or equivalently T_∞-tribes in the terminology of Butnariu and Klement [2], are families of fuzzy sets which are in fact σ-complete MV-algebras, where all MV-operations and $\wedge$ and $\vee$ are defined pointwisely. Tribes are important for representing σ-complete MV-algebras by a Loomis–Sikorski type theorem which says that every σ-complete MV-algebra is a σ-epimorphic image of some tribe of fuzzy sets, [1,4,15].

On the other hand, in [10], the authoresses used Markov kernels for describing some properties of quantum observables on σ-complete effect algebras and σ-MV-effect algebras. This case concerns processing information from quantum mechanical measurements which is measured using observables, analogues of measurable functions.

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Quantum mechanics is the most effective tool for description of the physical world. Recently new discoveries have been found for its applications in information and computation science [17]. As it was mentioned in the special issue of Information Sciences [14] dedicated to quantum structures, quantum information science is a new field of science and technology, combining and drawing on the disciplines of physical science, mathematics, computer science, quantum computing, and engineering. One of the sources of quantum information science comes from the quantum structures modeling, see [7].

For the Hilbert space quantum mechanics, bounded observables model quantum measurement and they are represented by Hermitian operators which form a po-group $\mathcal{B}(H)$ under the standard ordering of Hermitian operators. Here there are two models: (1) $\mathcal{P}(H)$, the system of orthogonal projectors on a Hilbert space H, and (2) $\mathcal{E}(H)$, the system of Hermitian operators that are between the zero operator and the identity operator. Then $\mathcal{P}(H)\subset\mathcal{E}(H)\subset B(H)$. Whereas $\mathcal{B}(H)$ is an antilattice, [11], $\mathcal{P}(H)$ is a complete orthomodular lattice, and $\mathcal{E}(H)$ has no lattice structure. Therefore, Olson [19] introduced a new order, which we call now the Olson order such that $\mathcal{B}(H)$ is a Dedekind complete lattice. Inspired by Olson, we have extended in [5] the Olson order for bounded observables.

Using similarities between a special kind of observables and Markov kernels, we define an analogue of the Olson order for the class of bounded Markov kernels associated with a tribe of fuzzy functions. To be able to define the Olson order, we show that there is a one-to-one relationship between Markov kernels and their spectral resolutions. We show that under the Olson order, the set of bounded Markov kernels associated with a tribe is a Dedekind σ-complete lattice. In addition, using spectral resolution, we will also define the sum $\uplus$ of two bounded Markov kernels. Then the set of bounded Markov kernels is a lattice-ordered Abelian semigroup. These results are main contributions of the paper. If we restrict the Olson order and the sum to the set of sharp bounded Markov kernels, then this set is even an Abelian σ-complete ℓ-group with strong unit. Finally, we show that our methods work also for the case when we study bounded Markov kernels associated with a T_s-tribe, where T_s is a Frank t-norm for $s\in(0,\infty)$. In addition, some illustrating examples are presented.

2. Preliminary notions

An *MV-algebra* is an algebra $M=(M;\oplus,\odot,^*,0,1)$ of signature $\langle 2, 2, 1, 0, 0\rangle$ such that the following holds for all $a, b, c\in M$

(i) $a\oplus b=b\oplus a$;
(ii) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$;
(iii) $a\oplus 0=a$;
(iv) $a\oplus 1=1$;
(v) $(a^*)^*=a$;
(vi) $a\oplus a^*=1$;
(vii) $0^*=1$;
(viii) $(a^*\oplus b)^*\oplus b=(a\oplus b^*)^*\oplus a$.

We recall (i) $a\le b$ iff there is $c\in M$ such that $a\oplus c=b$ and (ii) every MV-algebra is a distributive lattice, where $a\vee b=(a^*\oplus b)^*\oplus b$ and $a\wedge b=(a^*\vee b^*)^*$, $a, b\in M$. An MV-algebra M is a *σ-complete* MV-algebra if, for every sequence $\{a_n\}$ of elements of M, $\bigvee_n a_n\in M$. We note that MV-algebras are intervals in Abelian ℓ-groups with strong unit.

We remind that an Abelian ℓ-group $(G;+,0)$ written additively is a *partially-ordered group* (po-group for short) if G is endowed with a partial order $\le$ such that $a\le b$ implies $a+c\le b+c$ for each $c\in G$. If, in addition, $\le$ entails the lattice order, G is said to be a *lattice-ordered group* (ℓ-group for short). An element $u\in G$ such that $u\ge 0$ is said to be a *strong unit* if, given an element $g\in G$, there is an integer $n\ge 1$ such that $g\le nu$. A couple (G, u), where u is a fixed strong unit for G, is a *unital ℓ-group*.

An important class of MV-algebras consists of tribes, which are σ-complete MV-algebras of [0,1]-valued functions. We recall that a *tribe* on $\Omega\neq\emptyset$ is a collection $\mathcal{T}$ of fuzzy sets from $[0,1]^\Omega$ such that (i) $1\in\mathcal{T}$, (ii) if $f\in\mathcal{T}$, then $f':=1-f\in\mathcal{T}$, and (iii) if $\{f_n\}$ is a sequence from $\mathcal{T}$, then $\min\{\sum_{n=1}^\infty f_n,1\}\in\mathcal{T}$. A tribe is always a σ-complete MV-algebra, where all operations are defined pointwisely. We note that a tribe is a generalization of a σ-algebra of subsets of Ω because if $f_n=\chi_{A_n}$, where χ_{A_n} is the characteristic function of the set A_n, then $\min\{\sum_{n=1}^\infty\chi_{A_n},1\}=\chi_{\bigcup_n A_n}$. Moreover, if $\{f_n\}$ is a sequence of elements of $\mathcal{T}$, then $\inf_n f_n, \sup_n f_n\in\mathcal{T}$.

We notice that every tribe is a distributive lattice with the following strengthened forms of distributivity

$$f\wedge\left(\bigvee_n f_n\right)=\bigvee_n(f\wedge f_n),\quad f\vee\left(\bigwedge_n f_n\right)=\bigwedge_n(f\vee f_n)$$

for each $f, f_n\in\mathcal{T}$, $n\ge 1$, see [3].

3. Markov kernels associated with tribes

In the section, we define Markov kernels associated with a tribe. We show their relationship to the corresponding spectral resolutions which will allow us to introduce the Olson order of bounded Markov kernels such that this set becomes a Dedekind σ-complete lattice. In addition, we define the sum $\uplus$ of any two bounded Markov kernels such that the set of