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Binary generating set of the clone of idempotent aggregation functions on bounded lattices

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ABSTRACT

In a recent paper Botur et al. (2018) we have presented a generating set of the clone of idempotent aggregation functions on bounded lattices. As the main result we have shown that this clone is generated by certain ternary idempotent functions from which all idempotent aggregation functions of L can be obtained by usual term composition.

The aim of this paper is to present an essential improvement of the result above by presenting a new generating set of this clone. A bit artificial ternary functions are substituted here by natural (binary) lattice a-medians and certain binary characteristic functions. Consequently, the clone is generated by its binary part and the result strengthens the essential role of medians within all idempotent aggregation functions. Moreover, we will show that for an n-element lattice L, the upper bound of binary generators is $2n - 1$.

1. Introduction

Aggregation theory is a rapidly growing field of mathematics with roots in all domains where a merging of several inputs from a considered ordered scale into one output, reflecting some natural constraints, is considered. Its historical roots go back to Moscow papyrus (problem 14) and can be traced in the ancient Greece - recall several kinds of means, such as Heronian, for example. The major contribution in aggregation theory have delt with some real interval scales and they were summarized is monographs such as [1,2,8].

Only recently more abstract scales were considered, in particular lattice (poset) scales. For information sciences, and in particular for subjective decision problems, typical scales deal with bounded (distributive) lattices. Not going into details, among different papers dealing with aggregation functions acting on lattices, we recall e.g. the seminal papers [5,6], our recent papers [11–13], or the papers on nullnorms and uninorms on bounded lattices...

Particular classes of aggregation functions can be seen as special clones, ranging from the smallest one (all projections) to the biggest one (all aggregation functions of a considered lattice L). One special case is related to the class of all idempotent aggregation functions on L, and its subclass of idempotent lattice polynomials (i.e., Sugeno integrals, see [5,6]). Note

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that in the case of real intervals, idempotent aggregation functions are just monotone means, and they are indispensable in several domains related to unanimous decision making. The possible complexity of mentioned clones can be reduced when we look on their generating sets, i.e., subsets of aggregation functions from the considered clone such that their compositions generates all members of that clone. This problem was discussed for idempotent lattice polynomials on a distributive bounded lattice L in [5,6], see also [9,10], and for all idempotent aggregation functions on a bounded lattice L in [3]. Note that the generating set introduced in [3] has contained also quite artificial ternary aggregation functions and it was far from to be minimal. The main aim of this work is to improve earlier results concerning the generating sets of idempotent aggregation functions on lattices, where only binary generating idempotent aggregation functions will occur. We also show that our introduced generating sets are minimal.

2. Algebraic preliminaries

This paragraph is devoted to recalling the necessary concepts from universal algebra, for details we refer to standard monographs [4,17]. By an *algebra* we mean a pair $(X; F)$ consisting of a nonempty set X (called the *support* of the algebra) and the set F (possibly empty) of operations on X. For any n-ary operation $f \in F$ and any n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ we denote by $f(\mathbf{x})$ the evaluation of f in $\mathbf{x}$.

We call a nonempty subset $B \subseteq X$ a *subalgebra* of $(X; F)$ if for any n-ary operation $f \in F$ and for any n-tuple $\mathbf{x} \in B^n$ we have $f(\mathbf{x}) \in B$. It is evident that $(B; F)$ is again an algebra where the operations are those on X but restricted to B.

Further, given an algebra $(X; F)$, by the *direct square* of $(X; F)$ we mean an algebra (X^2, F) with the support being the Cartesian square X^2 of X, and the operations defined component-wise, i.e. for any n-ary $f \in F$ and any n-tuple $\big((x_{11}, x_{12}), \ldots, (x_{n1}, x_{n2})\big)$ of elements of X^2 we have

$$f\big((x_{11}, x_{12}), \ldots, (x_{n1}, x_{n2})\big) = \big(f(x_{11}, \ldots, x_{n1}), f(x_{12}, \ldots, x_{n2})\big).$$

We say that a k-ary function g on X *preserves the subalgebras of the direct square* (X^2, F) if for any subalgebra B of (X^2, F), whenever we have a $2 \times k$ matrix (b_{ij}) of elements of X all the columns of which belong to B, then so does the 2-tuple when applying g to its rows, i.e.,

$$\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}, \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}, \ldots, \begin{pmatrix} b_{1k} \\ b_{2k} \end{pmatrix} \in B \quad \Longrightarrow \quad \begin{pmatrix} g(b_{11}, b_{12}, \ldots, b_{1k}) \\ g(b_{21}, b_{22}, \ldots, b_{2k}) \end{pmatrix} \in B.$$

Clone theory turned out to be an extremely useful part of universal algebra with many applications in different areas of mathematics. The concept of a clone comes from that of a monoid in a sense that monoids of selfmaps of a set X form a composition-closed class. Abstracting from this elementary example we arrive to a general definition of a clone. For an overview of clone theory we refer to [7,14–16].

A *clone* on a set X is a set of (finitary) operations on X which contains all the projections on X and that is closed under the composition in the following sense.

For a set X, a positive integer $n \in \mathbb{N}$ and for any $i \le n$, the *ith n-ary projection* is for all $x_1, \ldots, x_n \in X$ defined by

$$p_i^n(x_1, \ldots, x_n) := x_i.$$

Composition is then formed as follows: given a k-ary operation f: $X^k \to X$ and k n-ary operations $g_1, \ldots, g_k : X^n \to X$, their composition is an n-ary operation $f(g_1, \ldots, g_k) : X^n \to X$ defined by

$$f\big(g_1, \ldots, g_k\big)(x_1, \ldots, x_n) := f\big(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\big), \tag{1}$$

for all $x_1, \ldots, x_n \in X$.

It can be easily checked that the composition is a usual product of selfmaps for $k = n = 1$. A set of functions which is closed under composition is said to be composition-closed. Hence the clones are composition-closed classes containing all projections.

Let us note that clones on a set X can be viewed equivalently as the term operations of algebras on X. This follows from the fact that algebraic terms include the projections and they are composed in the same way as functions in clones.

The set of all clones on X (ordered by set inclusion) has always the least and the greatest element, the least one called the *trivial clone* on X consisting just of all the projections, the greatest one called the *full clone* on X containing all the functions on X (denoted by $\mathcal{O}_X$).

For any subset $F \subseteq \mathcal{O}_X$ we always have the least clone $[F]$ on X containing F. It can be obtained as the intersection of all clones on X containing F and we call it the clone generated by F. If $\mathcal{C} = [F]$ for some finite set F, then $\mathcal{C}$ is said to be finitely generated.

Clones on a finite set X containing the so-called *near-unanimity function* are known to be finitely generated. Recall that for $n \ge 3$, an n-ary function f on X is called a *near-unanimity function* if

$$f(y, x, \ldots, x) = \cdots = f(x, \ldots, x, y, x, \ldots, x) = \cdots = f(x, \ldots, x, y) = x$$

for any $x, y \in X$, i.e., if any $n - 1$ of the n inputs coincide, the output of f takes the same value. In particular, 3-ary near-unanimity functions are called *ternary majority functions* on X.