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Residuated skew lattices

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ABSTRACT

Skew lattices are one of the most successful non-commutative generalizations of lattices. Motivated by the study of residuation on ordered structures and that of skew Boolean algebras and skew Heyting algebras, in this paper, we introduce the concept of residuated skew lattices as a new non-commutative version of residuated lattices. The axiomatization and localization for residuated skew lattices are obtained. Moreover, a characterization of residuation on skew lattices via adjointness is presented. Meanwhile, three special subvarieties: distributive residuated skew lattices, skew MTL-algebras and skew BL-algebras, are investigated. In particular, we show that every skew Heyting algebra is a reduct of a special residuated skew lattice.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth [18], provide algebraic semantics for certain non-classical logic calculi known as substructure logic [9]. Usually, a residuated lattice L is a (2, 2, 2, 2, 0, 0)-type algebra, written as $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, $(L, \otimes, 1)$ is a commutative monoid, and $(\otimes, \rightarrow)$, called "multiplication" and "residuum" respectively, forms an adjoint pair ($a\otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$). Corresponding to fuzzy logic, the operations $\otimes$ and $\rightarrow$ are considered as logic connectives "conjunction" and "implication". Residuated lattices and its related subvarieties, such as MV-algebras by Chang [14], BL-algebras by Hájek [19], R_0-algebras by Wang [30] and integral commutative l-monoids by Höhle [20], play important roles in fuzzy logic theory, providing algebraic frameworks for fuzzy logic and fuzzy reasoning [16,27,28,31].

Different forms of generalized residuated lattices are proposed and widely investigated. For examples, Zhou and Li [32] presented partial residuated structures as a common generalization of residuated lattices and lattice effect algebras in 2008. Ahamed and Ibrahim [2] proposed residuated weak lattices based on the weak lattices [1] in 2010. Using the partial operations, a conditionally residuated structure was defined by Chajda and Halaš [12] for representations of effect algebras in 2011. In 2013, Chajda and Kraňávek [13] introduced skew residuated lattices by changing the adjointness property. In this paper, we propose a new non-commutative version of residuated lattices based on skew lattices which are non-commutative lattices.

Non-commutative lattices were initially introduced in 1949 by Jordan [21], motivated by questions in quantum mechanics. Generally speaking, a non-commutative lattice is an algebra $(S, \vee, \wedge)$, where operations $\vee$ and $\wedge$ are both associative, idempotent and jointly satisfy a certain collection of absorption identities. Based on the investigations of the bands, a thorough classification of non-commutative lattices with respect to the Green's Relations is given by Laslo and Leech [23]. An

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important collection of non-commutative lattices called skew lattices were introduced by Leech [24] in 1989. A skew lattice is an algebra $(S, \vee, \wedge)$, where S is a nonempty set, both $\vee$ and $\wedge$ are binary operations on S satisfying associative, idempotent laws and two special absorption identities: $(a \vee (a \wedge b) = a = a \wedge (a \vee b)$ and $(a \wedge b) \vee b = b = (a \vee b) \wedge b)$. Skew lattices have shown their importance in quantum (computational) logic, multiple-valued logic [6,7], computer science [8,29], finance and economics [10,11], and so on.

Recently, based on skew lattices, skew Boolean algebras and skew Heyting algebras are introduced as a new non-commutative version of Boolean algebras and Heyting algebras [17,25]. The Stone duality and Priestley duality of skew Boolean algebras and distributive skew lattices are obtained by Kudryavtseva [22] and Cvetko-Vah [3,4]. Especially, skew Boolean algebras and skew lattices are closely related to the operations "override" and "update" in programming languages [5]. In this paper, we consider a skew version of residuated lattices in order to generalize both skew Boolean algebras and skew Heyting algebras, which we will call residuated skew lattices.

Boolean algebras and Heyting algebras are special residuated lattices. We know that every principal filter of a conormal skew lattice under the natural partial order is a lattice. Thus, both the definitions of skew Boolean algebras and skew Heyting algebras are given by forcing all principal filters to be classic algebras. In particular, we know that the operation "$\otimes$" is the same as "$\wedge$" in Boolean algebras and Heyting algebras. However, this is not the case for general residuated lattices. Hence the adjointness of "$\otimes$" and "$\rightarrow$" will be changed when using skew lattices to replace the lattice structures in these algebras. Actually, in the definition of skew Heyting algebras, the implication operator is obtained by defining it "locally" in $\uparrow u$ and the similar adjointness can be obtained by the natural quasi-order.

One easily sees that the adjointness is the core property in defining residuated lattices. Hence we want to keep a similar adjointness property when we generalize them to a skew version. Unlike Boolean algebras and Heyting algebras, the operations "$\otimes$" and "$\wedge$" are usually different in residuated lattices, this means that the operations "$\wedge$" and "$\rightarrow$" are not closely related and the lattice structure on a residuated lattice may have little influence on the corresponding implication operator. Moreover, if we define the operation "$\otimes$" locally just like "$\rightarrow$" on skew Heyting algebras, it will be hard to get a global operation "$\otimes$" satisfying the associative law. Hence, the adjointness under the natural quasi-order may not hold if we choose a similar method to define residuated skew lattices. In this paper, by defining the operation "$\otimes$" globally, we give an appropriate definition of residuated skew lattices, where the appropriateness of our definition will be justified by three different equivalent characterizations via adjointness, axiomatization and localization, respectively.

The paper is organized as follows: In Section 2, we introduce some basic concepts and results on skew lattices and residuated lattices. By introducing quasi-residuated skew lattices and monoidal skew lattices, the definition of residuated skew lattices is given and their properties are discussed in Section 3. In particular, we give the axiomatization and localization of residuated skew lattices and its join-bidistributive subvariety. We consider the connections between residuated skew lattices and skew Heyting algebras. The relations between the quotient residuated lattices and residuated skew lattices are also discussed. Section 4 is devoted to defining two special subvarieties of residuated skew lattices: skew MTL-algebras and skew BL-algebras, and to the discussions of their properties and axiomatizations.

2. Preliminaries

In this section, we recall some basic notions and properties of skew lattices and residuated lattices, which will be widely used in this paper.

2.1. Basic concepts of skew lattices

A **skew lattice** is an algebra $(S, \vee, \wedge)$, where S is a nonempty set, both $\vee$ and $\wedge$ are binary operations on S, named **join** and **meet** respectively, satisfying

SL_1: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
SL_2: $x \vee x = x$ and $x \wedge x = x$;
SL_3: $x \vee (x \wedge y) = x = x \wedge (x \vee y)$ and $(x \vee y) \wedge y = y = (x \wedge y) \vee y$.

The equations introduced above are known as the **associative laws**, the **idempotent laws**, and the **absorption laws**, respectively. Clearly, a lattice is a skew lattice satisfying the **commutative laws**:

L_4: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.

Similar to lattices, a skew lattice has a natural **partial order** "$\leq$" defined as follows:

$$y \leq x \Leftrightarrow x \vee y = y \vee x = x \text{ or equivalently, } x \wedge y = y \wedge x = y.$$

In a skew lattice, there is a natural **quasi-order**"$\preceq$" defined as follows:

$$y \preceq x \Leftrightarrow x \vee y \vee x = x \text{ or equivalently, } y \wedge x \wedge y = y.$$

The algebraic reducts $(S, \vee)$ and $(S, \wedge)$ of a skew lattice S are **regular bands**, with the properties:

$$x \vee y \vee z = x \vee z \text{ if } y \preceq x, z \text{ and } x \wedge y \wedge z = x \wedge z \text{ if } x, z \preceq y.$$