Information Sciences 430–431 (2018) 39–45

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

On generating of idempotent aggregation functions on finite lattices

Michal Botur[a,*], Radomír Halaš[a], Radko Mesiar[a,b], Jozef Pócs[a,c]

[a] Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc, 17. Listopadu 12, Olomouc 771 46, Czech Republic
[b] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského, Bratislava 1 11, 810 05, Slovakia
[c] Mathematical Institute, Slovak Academy of Sciences, Grešákova Košice, 6, 040 01, Slovakia

ARTICLE INFO

Article history:
Received 10 November 2016
Revised 15 May 2017
Accepted 16 November 2017
Available online 20 November 2017

MSC:
06B99

Keywords:
Idempotent aggregation function
Generating set
Finite lattice

ABSTRACT

In a recent paper [9] we proposed the study of aggregation functions on lattices via clone theory approach. Observing that aggregation functions on lattices just correspond to 0,1-monotone clones, as the main result of [9], we have shown that all aggregation functions on a finite lattice L can be obtained as usual composition of lattice operations $\wedge$, $\vee$, and certain unary and binary aggregation functions.

The aim of this paper is to present a generating set for the class of intermediate (or, equivalently, idempotent) aggregation functions. This set consists of lattice operations and certain ternary idempotent aggregation functions.

1. Introduction

Aggregation is a process when (usually numerical) data are merged in a single output. Mathematically, the process of aggregation is based on the concept of aggregation function.

The most natural examples of aggregation functions widely used in experimental sciences are means and averages (such as e.g. the arithmetic mean). These belong to a widely studied class of so-called internal aggregation functions, firstly mentioned by Cauchy already in 1821, with a huge variety of applications. Nowadays, aggregation functions are successfully applied in many different branches of science, we can mention e.g. social sciences, computer science, psychology etc.

As the process of aggregation somehow "synthesizes" the input data, aggregation functions cannot be arbitrary and have to fulfill some natural minimal requirements. This can be translated into the condition that the output value should lie in the same interval as the input ones, and the least and the greatest values should be preserved. Another widely accepted property of aggregation functions is that the output value should increase or at least stay constant whenever the input values increase.

The theory of aggregation functions is well developed in a case when the input (and, consequently, the output) values of these functions lie in a nonempty interval of reals, bounded or not. For details, we can refer the reader to the comprehensive monograph [2]. During several last years, the theory was enlarged to lattice-based data, i.e. when the input (as well as the

* Corresponding author.
E-mail addresses: michal.botur@upol.cz (M. Botur), radomir.halas@upol.cz (R. Halaš), radko.mesiar@stuba.sk (R. Mesiar), pocs@saske.sk (J. Pócs).

https://doi.org/10.1016/j.ins.2017.11.031
0020-0255/

output) data have a structure of a lattice. This more general approach allows us to work e.g. with data which are not linearly ordered or when information about the input data is incomplete.

Recall that a lattice is an algebra $(L; \vee, \wedge)$, where L is a nonempty set with two binary operations $\vee$ and $\wedge$ representing suprema and infima. Let us mention that lattice theory is a well established discipline of general algebra. There are several monographs on this topic, among them the most frequently used and quoted is the book by G. Grätzer, [3].

The formal definition of aggregation function is as follows:

An aggregation function on a bounded lattice L is a function A: $L^n \to L$ that

(i) is nondecreasing (in each variable), i.e. for any $\mathbf{x}, \mathbf{y} \in L^n$:

$$A(\mathbf{x}) \leq A(\mathbf{y}) \quad \text{whenever} \quad \mathbf{x} \leq \mathbf{y},$$

(ii) fulfills the boundary conditions

$$A(0, \ldots, 0) = 0 \quad \text{and} \quad A(1, \ldots, 1) = 1.$$

The integer n represents the arity of the aggregation function.

The set of all n-ary aggregation functions can be naturally ordered via component-wise ordering, i.e., if A and B are n-ary aggregation functions, then $A \leq B$ provided $A(\mathbf{x}) \leq B(\mathbf{x})$ for all $\mathbf{x} \in L^n$. Note that with respect to such ordering the set of all n-ary aggregation functions forms a bounded lattice. In the sequel, when referring to the order of aggregation functions, we have in mind this component-wise ordering.

Clearly, one of the central problems in aggregation theory is to give their construction methods. Much of work has already been done in this direction which fact can be easily demonstrated by the extensive literature including book chapters, we refer the reader e.g. to a standard monograph [2]. It can be recognized that methods like composed aggregation, weighted aggregation, forming ordinal sums etc., look quite different and each of them relies on a very specific approach. In a classical case, the idea is based on standard arithmetical operations on the real line and fixed real functions.

From the point of view of universal algebra, aggregation functions on a lattice L form a clone (or, equivalently, a composition-closed set of functions containing the projections), the so-called aggregation clone of L. Recall that the clone theory is a very well established discipline of universal algebra. In principle, it deals with function algebras and its development was mostly initiated by studies in many-valued logic. For details, we refer the reader to the books [9,11] or to the overview paper [10].

The clone of aggregation functions on L forms a subclone of the so-called monotone clone of L consisting of all functions on L preserving the lattice order. It is well-known that for a finite lattice L, any clone on L containing the lattice operations is finitely generated. Although generating sets of monotone clones on finite lattices can be found e.g. in [11], these cannot be directly used for generating of any of its proper subclones. In [8], for any n-element lattice, we presented explicitly at most $(2n+2)$-element generating set of aggregation functions, consisting of lattice binary operations, at most n unary functions and at most n binary aggregation functions. Consequently, we have shown that any aggregation function on L arises as the usual term composition of the above mentioned set of generating functions. Moreover, contrary to the case of the monotone clone, we have shown that each generating set of the aggregation clone containing the lattice operations, must posses at least one another operation of arity higher than one. Let us mention that for the classical case when $L = [0, 1]$ is the unit interval of reals, another generating set of functions has been presented in [6]. For generating other types of aggregation functions we refer to [1].

Although the above mentioned generating set allows us to construct the set of all aggregation functions on a bounded lattice L, there is a question how to generate some of its important subclasses being composition-closed. For example, all intermediate (or, equivalently, idempotent) aggregation functions form such a class. Since there is only one unary idempotent function, namely the identity function, the generating set described in [8] cannot be applied. The aim of this paper is to present a generating set for the class of all idempotent aggregation functions, members of which will be itself idempotent aggregation functions. In fact, as the main result we will show that certain ternary idempotent aggregation functions together with the lattice operations form a generating set. We expect the extension of our results for certain important composition-closed subclasses of idempotent aggregation functions, especially the class of Sugeno integrals [4,5] on bounded distributive lattices.

2. Idempotent aggregation functions

For a set X and a natural number n, denote by $O_n(X)$ the set of all n-ary functions on X, i.e. the mappings f: $X^n \to X$. We put $O(X) = \bigcup_{n=1}^{\infty} O_n(X)$.

Let X be a set and $n \in \mathbb{N}$ be a positive integer. For any $i \leq n$, the *ith n-ary projection* is for all $x_1, \ldots, x_n \in X$ defined by

$$p_i^n(x_1, \ldots, x_n) := x_i. \tag{1}$$

Composition of functions on X forms from one k-ary function f: $X^k \to X$ and k n-ary functions $g_1, \ldots, g_k : X^n \to X$, an n-ary function $f(g_1, \ldots, g_k) : X^n \to X$ defined by

$$f\big(g_1, \ldots, g_k\big)(x_1, \ldots, x_n) := f\big(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\big), \tag{2}$$

for all $x_1, \ldots, x_n \in X$.