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Lattice-theoretic contexts and their concept lattices via Galois ideals

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ABSTRACT

This paper introduces a concept of lattice-theoretic contexts as well as their concept lattices. A lattice-theoretic context is a triple (G, M, I) with two complete lattices G, M and their Galois ideal I. A lattice-theoretic context and its concept lattice are a common generalization of classical FCA, Pócs's formal fuzzy context, one-sided concept lattices, generalized concept lattices and L-fuzzy concept lattices (with hedges). When the lattices G, M are completely distributive, a reduction of the relation I in the lattice-theoretic context (G, M, I) can be obtained. Related algorithms to construct concept lattices of L-fuzzy contexts considered as lattice-theoretic contexts are presented. In the case of L being a completely distributive lattice, we can reduce the number of elements (objects or/and attributes) before computing the whole concept lattice. Then the related algorithm has lower complexity.

1. Introduction

Formal concept analysis (FCA) is a mathematical theory, originally proposed by Wille [42], which provides theoretical support for conceptual data analysis and knowledge discovery. The aim of FCA is to build a hierarchical structure of clusters from a dataset of objects together with their attributes (also called properties) and to use this conceptual hierarchy as a knowledge discovery and a representation tool.

Given such a dataset (called a formal context), two important Galois connections can be defined between the powerset of objects and that of attributes. The first one is an antitone Galois connection, originated by Wille [42] and studied extensively in the literatures. The second one to define conceptual structures by addressing the objectives of knowledge processing is provided by the modal style operators framework. Based on two approximation operators, which form an isotone Galois connection, Gediga and Duntsch [15] defined a new type of concept lattices, i.e., the attribute-oriented concept lattices. Using sets of objects instead of attributes, Yao [43] introduced in a similar way the notion of object-oriented concept lattices. Besides, many other different approaches to concept lattices have been introduced in recent years [22,40,44].

Many mathematical structures related to sets can be extended to certain structures of powersets and consequently be studied as ordered structures. For example,

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(1) For every topological space $(X, \mathscr{T})$, the pair $(\mathscr{T}, \subseteq)$ is a frame, that is, a complete lattice satisfying the infinite distributive law of binary meets over arbitrary joins. This structure indicates that we can study topology directly on $\mathscr{T}$ itself by forgetting the point set X. Such an approach gives rise to the so-called pointfree topology theory or locale theory [21] with frames as the main objects.
(2) For a graph $G=(V, E)$ with V as the vertex set and E the edge set, suppose that $\mathcal{I}$ is the family of subsets of E such that every member does not contain any cycle of G. The structure $(E, \mathcal{I})$ is called a graphic matroid [41], which forgets the point set E in G. We can further study pointfree structure $(\mathcal{I}, \subseteq)$ as an ordered structure, which gives rise to the supermatroid theory [13].
(3) The notion of ideals is an important tool to study the structure of a given algebra. For example, given a C^*-algebra $(A, *)$ (also called a von Neumann algebra) [1], the set of closed right ideals $CRI(A)$ of $(A, *)$ forms a right-sided and idempotent quantale [35]. Some properties of the C^*-algebra can be reflected in this quantale. So by forgetting the point set A, $CRI(A)$ can be considered as an ordered structure of the C^*-algebra $(A, *)$, which gives rise to the quantale theory [35].

For a formal context (X, Y, R), we also could consider it as a structured set. Its concept lattice $\mathfrak{B}(X, Y, R)$ (which will be written as $\mathfrak{B}_c(X, Y, R)$ in the rest of this paper) is a subset of the product of the powersets 2^X and 2^Y. A natural question is that, can we define a context directly on the powersets 2^X and 2^Y? Or in other words, can we establish a concept lattice based on a relation from 2^X to 2^Y?

For every formal context (X, Y, R), two fundamental operators $^{\uparrow}: 2^X \longrightarrow 2^Y$ and $^{\downarrow}: 2^Y \longrightarrow 2^X$ are defined as

$$A^{\uparrow}=\{m \in M|\ \forall a \in A, (a, m) \in R\},$$

$$B^{\downarrow}=\{g \in G|\ \forall b \in B, (g, b) \in R\}.$$

It is known to all that

$$A \subseteq B^{\downarrow} \text{ iff } A \times B \subseteq I \text{ iff } B \subseteq A^{\uparrow}\ (\forall A \subseteq G, \forall B \subseteq M).$$

Thus $((\ )^{\uparrow}, (\ )^{\downarrow})$ is a Galois connection between $(2^X, \subseteq)$ and $(2^Y, \subseteq)$ (Proposition 10 in [14]). The concept lattice of (X, Y, R) is

$$\mathfrak{B}_c(X, Y, R)=\{(A, B) \in 2^X \times 2^Y \mid A=B^{\downarrow}, B=A^{\uparrow}\}.$$

Every member in $\mathfrak{B}_c(X, Y, R)$ is called a *concept* of the formal context (X, Y, R).

It is easy to see that (A, B) is a concept iff $A \times B$ is a maximal rectangle in R, where neither A nor B can be enlarged anymore. This property helps us to study the context $(2^X, 2^Y, I_R)$, where I_R is the set of all rectangles in R, which will be called a *power context* in this paper. Frankly speaking, the power context $(2^X, 2^Y, I_R)$ has no more information than the original context (X, Y, R), and hence they will have the same (or isomorphic) concept lattices. But it motivates us to formulate the power context $(2^X, 2^Y, I_R)$ by a triple (G, M, I) in lattice-theoretic setting, where G, M are complete lattices and I is a Galois ideal of them.

Let us call such a triple (G, M, I) a *lattice-theoretic context*. In fact, on one hand, a lattice-theoretic context can be considered as a pointfree version of a formal context, just like locales as pointfree version of topological spaces [21]; on the other hand, it can still be considered as a classical formal context with some lattice structures and properties. Hence, there will be two concept lattices related to every lattice-theoretic context. Then there comes a natural question that

Question 1. What is the relation between these two lattice-theoretic contexts as well as their concept lattices?

There are several approaches and methods of how to construct fuzzy concept lattices. We mention the approach of Bělohlávek [2–4] based on logical framework of complete residuated lattices, the work of Georgescu and Popescu to extend this framework to non-commutative logic [16–18], and the approach of Krajči [25] and Popescu [33]. There are also other approaches generalizing the previous ones, for which we mention the approach of Medina et al. dealing with multi-adjoint concept lattices [28,29], the work of Jaoua and Elloumi on Galois lattices of real relations [20], and the paper on variable threshold concept lattices by Zhang et al. [44]. Most of these approaches can be characterized as searching for a Galois connection between powersets of complete lattices. Hence, a second question arises.

Question 2. Can we formulate those different kinds of fuzzy concept lattices by lattice-theoretic concept lattices?

The idea to study FCA in lattice-theoretic setting has been carried out by Krajči [25] and Pócs [32], therein generalized concept lattices and formal fuzzy contexts are proposed and studied, respectively. Then there comes a third question that

Question 3. What are the relations between the lattice-theoretic contexts and the above-mentioned contexts?

The aim of this paper is to introduce a concept of lattice-theoretic contexts as well as their concept lattices, and then answer Questions 1–3. In Section 2, we will recall some preliminaries which will be used throughout this paper. In Section 3, we will study the power contexts induced by formal contexts, which is a hint for the concept of lattice-theoretic contexts. In Section 4, we will study lattice-theoretic contexts and the related fundamental theorem. It will be shown that for every lattice-theoretic context its concept lattice in the classical sense is isomorphic to that in the lattice-theoretic sense. A comparison among lattice-theoretic contexts, formal fuzzy contexts and L-fuzzy concept lattices will be made. We will show that lattice-theoretic contexts are a pointfree version of formal fuzzy contexts, and will formulate L-fuzzy concept lattices into lattice-theoretic setting in a direct way. We will also study the reduction of lattice-theoretic contexts when the lattices are