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Continuous additive generators of continuous, conditionally cancellative triangular subnorms

Andrea Mesiarová-Zemánková*

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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ABSTRACT

The continuous, conditionally cancellative t-subnorms that posses a continuous, additive generator are discussed. Conditions for a continuous, conditionally cancellative t-subnorm to have a continuous, additive generator are described. Constructions of corresponding additive generators are also shown.

1. Introduction

The (left-continuous) t-norms and their dual t-conorms have an indispensable role in many domains such as probabilistic metric spaces [13], fuzzy logic [3], fuzzy control [14], non-additive measures and integrals [11] and others. More details about t-norms and their applications can be found in [2,5]. Recall that although the structure of t-norms and of left-continuous t-norms is not known, continuous t-norms were characterized as ordinal sums of t-norms with continuous additive generators [5,6,10]. The basic stones for construction of t-norms via the ordinal sum construction are t-subnorms introduced by Jenei in [4] (see also [5,8]). Therefore the knowledge of the structure of t-subnorms plays a major role in the description of the structure of t-norms. Similarly as in the case of t-norms, also in the case of t-subnorms, the possibility to represent a t-subnorm by means of an additive generator, i.e., to represent a two-place function by means of a function of one variable, means an advantage of reduced complexity. On the other hand, if an additive generator is an isomorphism, then it is evident that the corresponding t-subnorm is an isomorphism of the (truncated) addition on some interval $[a, b]^2 \subseteq [0, \infty]^2$. A triangular norm can be represented by means of a continuous additive generator if and only if it is continuous and Archimedean and construction of the respective additive generator based on results of Aczél [1] can be found in [5]. In [7] additive generators of continuous, cancellative t-subnorms were studied. In this paper we would like to extend these considerations also to continuous, conditionally cancellative t-subnorms.

In Section 2 we will recall basic notions and results. We will study t-subnorms that posses continuous strictly monotone additive generators in Section 3 and in Section 4 we will focus on additive generators which are not strictly monotone. We give our Conclusions in Section 5.

* Tel.: +421 2 57 510 501.
E-mail address: zemankova@mat.savba.sk

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2. Basic notions and results

We will start with several useful definitions (see [2,5]).

Definition 1.

(i) A binary function $T:[0,1]^2 \longrightarrow [0,1]$ is a *t-norm* if it is commutative, associative, non-decreasing in both variables and 1 is its neutral element.
(ii) A binary function $C:[0,1]^2 \longrightarrow [0,1]$ is a *t-conorm* if it is commutative, associative, non-decreasing in both variables and 0 is its neutral element.

Due to the associativity, n-ary form of any t-norm (t-conorm) is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm C by the equation

$$C(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

A t-norm T is called *Archimedean* if for all $x, y \in]0,1[$ there exists an $n \in \mathbb{N}$ such that $x_T^{(n)} < y$, where $x_T^{(n)} = T(\underbrace{x, T(x, \ldots)}_{n\text{-times}})$.

A continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm C) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($C(x,y) = 1$).

Next we recall the result of [5, Theorem 5.1].

Theorem 1. *For a function $T:[0,1]^2 \longrightarrow [0,1]$ the following are equivalent:*

(i) *T is a continuous Archimedean t-norm.*
(ii) *T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t:[0,1] \longrightarrow [0,\infty]$, which is uniquely determined up to a positive multiplicative constant, such that for all $(x,y) \in [0,1]^2$ there is*

$$T(x,y) = t^{-1}(\min(t(0), t(x)+t(y))).$$

Similar result can be shown for additive generators of continuous t-conorms. Each additive generator t of a strict t-norm fulfills $t(0) = \infty$ and each additive generator t^* of a nilpotent t-norm fulfills $t^*(0) < \infty$.

In a more general situation of non-continuous t-norms also non-continuous additive generators can be assumed. However, here the use of the pseudo-inverse function is needed.

Definition 2.

(i) Let $t:[0,1] \longrightarrow [0,\infty]$ be a non-increasing function. Then the function $t^{(-1)}:[0,\infty] \longrightarrow [0,1]$ given by

$$t^{(-1)}(x) = \sup\{y \in [0,1] \mid t(y) > x\}$$

is called the *pseudo-inverse* of t.
(ii) A strictly decreasing function $t:[0,1] \longrightarrow [0,\infty]$, $t(1) = 0$, is called an *additive generator* of a t-norm $T:[0,1]^2 \longrightarrow [0,1]$ if

$$T(x,y) = t^{(-1)}(t(x)+t(y))$$

for all $(x,y) \in [0,1]^2$.

Note that if we would relax the strict monotonicity condition of the additive generator of a t-norm then the neutral element 1 would be lost. Several interesting results on additive generators of non-continuous t-norms can be found in [15]. Next we define a t-subnorm (see [4]) and its dual t-superconorm.

Definition 3.

(i) A binary function $S:[0,1]^2 \longrightarrow [0,1]$ is a *t-subnorm* if it is commutative, associative, non-decreasing in both variables and $S(x,y) \leq \min(x,y)$ for all $(x,y) \in [0,1]^2$.
(ii) A binary function $M:[0,1]^2 \longrightarrow [0,1]$ is a *t-superconorm* if it is commutative, associative, non-decreasing in both variables and $M(x,y) \geq \max(x,y)$ for all $(x,y) \in [0,1]^2$.

Between t-subnorms and t-superconorms there is the same duality as between t-norms and t-conorms. Thus all results which we will show for t-subnorms can be obtained immediately also for t-superconorms. It is evident that each t-norm (t-conorm) is also a t-subnorm (t-superconorm). Therefore to emphasize that a t-subnorm is not a t-norm we will call such a t-subnorm *proper*.

Since 1 need not to be a neutral element of a t-subnorm, in the case of t-subnorms we can assume also additive generators which are not strictly monotone, and also the condition $t(1) = 0$ can be relaxed.