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Conditional belief functions as lower envelopes of conditional probabilities in a finite setting

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ABSTRACT

The aim is to provide a characterization of full conditional measures on a finite Boolean algebra, obtained as lower envelope of the extensions of a full conditional probability defined on another finite Boolean algebra. Such conditional measures are conditional belief functions defined by means of a generalized Bayesian conditioning rule relying on a linearly ordered class of belief functions. This notion of Bayesian conditioning for belief functions is compared with other well-known conditioning rules by looking for those conditional measures that can be seen as lower conditional probabilities.

1. Introduction

Inferential processes play a central role in decisions under uncertainty and in knowledge acquisition. In particular, the adopted notion of conditioning is crucial in drawing inferences as it affects the final result of inferential processes: prototypically, conditional probability can be seen as a derived notion, as in the usual Kolmogorovian setting [39] or, in the more general context due to de Finetti, Dubins and Rényi [23,28,49], as a primitive concept defined axiomatically.

The probabilistic framework is the most common choice for dealing with uncertainty, even if sometimes the decision maker has only an incomplete, partial, imprecise probabilistic information and so it is suitable to accept a more general view than the probabilistic one (see, e.g., [10,24,25,32,58]). This is particularly interesting when it is infeasible to determine a unique probability model (e.g., when misclassified or latent variables are present) since the available information refers to a set of events different from those of interest.

As a paradigmatic example, consider a pension system based on the partition $\{E_1, \ldots, E_n\}$ generated by the age and the pension contributions of the last ten years, and suppose that this system needs to be modified into a new one in view of a legal reform. The new system involves the partition $\{F_1, \ldots, F_m\}$ generated by the age, the years of pension contributions and all the amount of pension contributions. Given a probability distribution P (related to a certain population) on the events $\{E_1, \ldots, E_n\}$, in order to draw inferences related to the new system we need to extend the probabilistic knowledge to the events $\{F_1, \ldots, F_m\}$.

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These situations naturally fit into de Finetti's theory of coherent conditional probability extensions, where, in order to avoid the loss of information and the introduction of arbitrary choices by the decision maker, we need to consider the whole set of coherent extensions summarized by the related lower and upper envelopes. This implies that in such situations it is unavoidable to manage a set of probabilities [58].

It is well-known that the lower [upper] envelope of the coherent extensions is generally a capacity, not necessarily 2-monotone [2-alternating] (see, for instance, [12,58] and Example 1 in [4]). Nevertheless, starting from a probability on an algebra the lower and upper envelopes of the extensions to another algebra are, respectively, a belief and a plausibility function (see [8,21,24,27,33,58]). Moreover, under specific (logical and numerical) constraints, one can obtain, besides a belief function, also a k-order additive measure [37,41] or even a necessity measure (see [9,10,20,30,42]).

The quoted capacities can be taken themselves as a framework for modeling uncertainty, nevertheless for them a commonly accepted notion of conditioning is still lacking even if many proposals have been addressed. Also the semantic of the different notions of conditioning can differ, in fact, for example, some are related to the restriction into a state of information (see, e.g., [18]) or to the combination of the evidence such as the Dempster rule [24].

Conditioning for capacities is usually faced emulating what is done for conditional probability. Hence, the existing proposals can be essentially divided into two classes: those introducing a conditional measure as a derived notion obtained, through a suitable conditioning rule, from an unconditional measure [19,24,26,30,35,50,53], and those presenting a conditional measure as a primitive concept, that is a function of two variables satisfying a set of axioms [1,6,11,13,14,59]. A distinguished notion of the first group is the Bayesian conditioning rule [24,33,38,43,55]: it defines (when it is possible) the conditional measure as the lower envelope of the conditional probabilities (in the Kolmogorovian sense) computed with the set of probabilities dominating the non-additive monotone measure. Note that generally this definition avoids conditioning to events with zero probability, nevertheless, dealing with a lower envelope, the problem of conditioning to events with zero lower probability becomes crucial. When the upper probability of a conditioning event H is greater than zero (but its lower probability is zero) a way to overcome the problem is the regular extension [58]. This consists in computing the lower envelope on a conditional event $E|H$ by using only a subset of the dominating probabilities (those positive on H).

Among the non-additive monotone measures discussed above, that are interesting *per se*, we restrict to belief functions since they are obtained as the lower envelope of the extensions of a probability defined on a Boolean algebra [4,33] and so they can be seen as restrictions of an inner measure, moreover, they are related also to rough sets and the computation of inner and outer approximations [15,16,18,45]. Then, we draw some conclusions for consonant belief functions, i.e., necessity measures.

Given a full conditional probability P defined on the finite Boolean algebra $\mathcal{A}$ (i.e., defined on $\mathcal{A} \times \mathcal{A}^0$ with $\mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$), we characterize the lower envelope of the coherent conditional probability extensions of P to another algebra. This leads to full Bayesian conditional (B-conditional) belief functions, that are a class of conditional measures generalizing conditional belief functions obtained by means of the Bayesian conditioning rule. Full B-conditional belief functions are shown to be characterized by a linearly ordered class of belief functions satisfying a covering condition, through a generalized Bayesian conditioning rule. Moreover, they are coherent conditional lower probabilities, i.e., they are the lower envelope of all the coherent conditional probability extensions of a full conditional probability defined on another algebra (see Section 3). The above hypothesis of fullness of the initial conditional probability (i.e., assuming $\mathcal{A} \times \mathcal{A}^0$ for its domain) is fundamental in order to get a full B-conditional belief function, as shown in Section 4 where the extension of a conditional probability (not necessarily full) is investigated and a closed form expression for the lower envelope of the coherent extensions is provided.

The aim of Section 5 concerns the comparison of conditioning rules, and the main question is the following one: do the axiomatic definitions of conditional belief functions "mimicking" the axiomatic definition of conditional probability give rise to a lower conditional probability? We show that none of such definitions determines, in general, a conditional measure which is a lower conditional probability. More in detail, such measures cannot be obtained as lower envelope of any subclass of coherent extensions of a full conditional probability. We also consider, in particular, full B-conditional belief functions generated by a linearly ordered class of necessity measures, obtaining a notion of Bayesian conditioning for necessity measures.

2. Conditional probability and its enlargement

Let $\mathcal{A}$ be a Boolean algebra of *events* E's, and denote with $(\cdot)^c$, $\vee$ and $\wedge$ the usual Boolean operations of negation (contrary), disjunction and conjunction, respectively. Consider also on $\mathcal{A}$ the partial order of implication $\subseteq$, where $E \subseteq F$ if and only if $E \vee F = F$. The *sure event* Ω and the *impossible event* $\emptyset$ coincide, respectively, with the top and bottom elements of $\mathcal{A}$.

A *conditional event* $E|H$ is an ordered pair of events (E, H) with $H \neq \emptyset$. In particular, as is well-known, any event E can be identified with the conditional event $E|\Omega$. In what follows we will mainly deal with sets $\mathcal{A} \times \mathcal{H}$ of conditional events, where $\mathcal{H} \subseteq \mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$ is an *additive class* (i.e., a set of events closed under finite disjunctions).

An arbitrary set of conditional events $\mathcal{G} = \{E_i|H_i\}_{i\in I}$ can always be embedded into a minimal set $\mathcal{A} \times \mathcal{H}$, where $\mathcal{A} = \langle\{E_i, H_i\}_{i\in I}\rangle$ is the Boolean algebra generated by $\{E_i, H_i\}_{i \in I}$ and $\mathcal{H}$ is obtained closing $\{H_i\}_{i \in I}$ under finite disjunctions. In the following such set $\mathcal{A} \times \mathcal{H}$ is denoted as $\langle\langle\mathcal{G}\rangle\rangle$.