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Extension of bivariate means to weighted means of several arguments by using binary trees

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ABSTRACT

Averaging aggregation functions are valuable in building decision making and fuzzy logic systems and in handling uncertainty. Some interesting classes of averages are bivariate and not easily extended to the multivariate case. We propose a generic method for extending bivariate symmetric means to n-variate weighted means by recursively applying the specified bivariate mean in a binary tree construction. We prove that the resulting extension inherits many desirable properties of the base mean and design an efficient numerical algorithm by pruning the binary tree. We show that the proposed method is numerically competitive to the explicit analytical formulas and hence can be used in various computational intelligence systems which rely on aggregation functions.

1. Introduction

Aggregation functions play an important role in many applications including decision making, fuzzy systems and image processing [1–3]. Averaging functions, whose prototypical examples are the arithmetic mean and the median, allow compensation between low values of some inputs and high values of the others. Such functions are also important for building decision models in weighted compensative logic [4], where the concepts of Generalized Conjunction/Disjunction (GCD) play a role [5,6].

The major class of averaging functions is the class of weighted quasi-arithmetic means (QAM) [1,2]. These functions are well studied and are convenient to work with as they have a natural definition for any number of arguments. Yet there are many other means, that often generalize quasi-arithmetic means, which are defined with respect to two arguments only, and do not offer a straightforward multivariate extension. A basic example here is the logarithmic mean [7]

$$\mathcal{L}(x, y) = \frac{x - y}{\ln x - \ln y},$$

which belongs to a rather broad class of Cauchy means. Cauchy means are defined with respect to two differentiable generating functions g and h such that $h' \neq 0$, by the use the Cauchy mean value theorem.

In turn, Cauchy means have several prominent subclasses, such as the Lagrangian mean, the generalized logarithmic mean, Stolarsky means, and also quasi-arithmetic means. Frequently used members include the already mentioned logarithmic means, the identric mean and the Stolarsky means.

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Another class of bivariate means are the neo-Pythagorean means [7], which are defined in terms of ratios between the inputs and the outputs. Here again, no obvious multivariate extension is present. One particular case is the Heronian mean recently studied in several works [8–10], which can be "naturally" extended to the multivariate case in several different ways.

There have been numerous attempts to extend various classes of bivariate means to n variables, and to incorporate the importance weights into this process. In particular, the logarithmic mean and the Stolarsky means were extended in different ways [11–15], notably through their representations as definite integrals of certain functions. Another way of extending the logarithmic mean and some related functions is through a series representation [12]. Finally, the divided differences approach is also useful in this respect [12].

The mentioned extensions, some of which we briefly present in the sequel, are undoubtedly useful for various applications, but they are not sufficiently general as to apply them to an arbitrary bivariate mean, without using its specific properties or alternative representations. Such a generic approach for extending bivariate means would be valuable for applications in decision making, so that alternative aggregators could be tried and added to the arsenal of aggregation tools.

In this contribution we propose one generic approach for extending the bivariate means and incorporating weighting vectors based on repetitive application of the given bivariate function, originally reported in [16]. It does not require knowledge of the properties of the bivariate function or its alternative representations, and is not based on an analytic formula but on an efficient computational procedure.

Attempts using repetitive application of the bivariate functions to generate multivariate ones have been made in the past. Of course, associativity of the triangular norms and conorms, as well as uninorms and nullnorms allows their straightforward extension to the multivariate case [1,2].

Recursive means were presented by Cutello and Montero in [17,18] in the framework of OWA functions, and later used in [19]. These authors used a sequence of bivariate functions f_2^n, $n = 2, 3, \ldots$ and the relation

$$f_n(x_1, \ldots, x_n) = f_2^n(f_{n-1}(x_1, \ldots x_{n-1}), x_n),$$

where f_n denotes an n-variate function. In particular, taking $f_2^n(x, y) = \frac{(n-1)x+y}{n}$ we obtain the n-variate arithmetic mean f_n. Weighted quasi-arithmetic means can be generated in a similar way.

The approach we present here is also a recursive application of the bivariate function but in a different order, by constructing a binary tree with a suitable number of levels, where at each node the bivariate function is applied to the arguments of the child nodes. By using the idempotency of the means, we prune this tree to design a computationally efficient procedure. On the other hand, we are able to incorporate the weighting vectors by repeating the arguments as needed, following the approach of Calvo et al. [20].

Our binary tree approach is generic in terms of the starting bivariate idempotent function being used, but it is not exact, in the sense that it is aimed at *approximating* a weighted multivariate mean (with any desired accuracy). Indeed, the binary tree will not reproduce exactly the weighting vectors with *irrational* coefficients, or coefficients that do not have finite binary representation (e.g., $\mathbf{w} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$), in a finite number of iterations. We argue, however, that for computational purposes this is not inferior than even the explicit formulas: after all, all weighting vectors have finite binary representation in machine arithmetic, which we can match *exactly*.

We would like to emphasize that unlike the existing multivariate extensions which are custom-built for a specific mean, our approach is applicable to an arbitrary bivariate symmetric mean, transparent to the user, and automatically preserves some very useful properties of that bivariate mean. The availability of a computationally efficient procedure for evaluating the multivariate weighted mean together with the generality of the proposed construction makes our approach broadly applicable in practice.

The remainder of this article is structured as follows. In Section 2, we provide the necessary mathematical foundations which we rely on in the subsequent sections. We discuss multivariate extension of several means in Section 3. In Section 4 we provide our main definitions and study the fundamental properties of our construction. In Section 5 we present the computational algorithms and discuss their complexity. We illustrate the efficiency of the algorithms numerically in Section 6. Our conclusions are presented in Section 7.

2. Preliminaries

Consider now the following definitions adopted from [1–3,21]. Let $\mathbb{I} = [0, 1]$, although other intervals can be accommodated easily.

Definition 1. A function $f : \mathbb{I}^n \to \mathbb{R}$ is **monotone** (increasing) if $\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n, \mathbf{x} \le \mathbf{y}$ then $f(\mathbf{x}) \le f(\mathbf{y})$, with the vector inequality understood componentwise.

Definition 2. A function $f : \mathbb{I}^n \to \mathbb{I}$ is **idempotent** if for every input $\mathbf{x} = (t, t, \ldots, t), t \in \mathbb{I}$ the output is $f(\mathbf{x}) = t$.

Definition 3. A function $f : \mathbb{I}^n \to \mathbb{I}$ is **averaging** if for every $\mathbf{x}$ it is bounded by $\min(\mathbf{x}) \le f(\mathbf{x}) \le \max(\mathbf{x})$.

Averaging functions are necessarily idempotent, and monotone increasing idempotent functions are averaging. The term *mean* is often used synonymously with averaging, although there are means that are not necessarily monotone increasing (e.g., Bajraktarević means, see [7,22,23]) and in principle may not be averaging (although they are idempotent). The means referred to in this paper are monotone increasing, and hence are averaging.