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# A review of the relationships between implication, negation and aggregation functions from the point of view of material implication

Q1 A. Pradera[a,*], G. Beliakov[b], H. Bustince[c], B. De Baets[d]

[a] *Departamento de Ciencias de la Computación, Arquitectura de Computadores, Lenguajes y Sistemas Informáticos y Estadística e Investigación Operativa, Universidad Rey Juan Carlos, Calle Tulipán s/n, Móstoles, Madrid 28933, Spain*
[b] *School of Information Technology, Deakin University, 221 Burwood Hwy, Burwood 3125, Australia*
[c] *Departamento de Automática y Computación, Universidad Pública de Navarra, Campus Arrosadia s/n, Pamplona 31006, Spain*
[d] *KERMIT, Department of Mathematical Modelling, Statistics and Bioinformatics, Ghent University, Coupure links 653, 9000 Gent, Belgium*

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ABSTRACT

Implication and aggregation functions play important complementary roles in the field of fuzzy logic. Both have been intensively investigated since the early 1980s, revealing a tight relationship between them. However, the main results regarding this relationship, published by Fodor et al. in the 1990s, have been poorly disseminated and are nowadays somewhat obsolete due to the subsequent advances in the field. The present paper deals with the translation of the classical logical equivalence $p \rightarrow q \equiv \neg p \vee q$, often called material implication, to the fuzzy framework, which establishes a one-to-one correspondence between implication functions and disjunctors (the class of aggregation functions that extend the Boolean disjunction to the unit interval). The construction of implication functions from disjunctors via negation functions, and vice versa, is reviewed, stressing the properties of disjunctors (respectively, implication functions) that ensure certain properties of implication functions (disjunctors).



## 1. Introduction

*Aggregation functions*, which perform the combination of several inputs into a single output, are successfully used in many practical applications, and the interest in them is unceasingly growing (see, e.g., the recent monographs on the topic [7,14,30,56]). In turn, *implication functions*, which are a means for generalizing the classical Boolean implication to the fuzzy setting, have proved to be essential in many different fields, ranging from approximate reasoning and fuzzy control to fuzzy mathematical morphology and image processing (see the surveys [3,42] and the monographs [4,5] for details and appropriate references). Aggregation and implication functions appear to have a close relation, mainly realized via *negation functions*, which model the logical negation within the fuzzy framework.

Many different methods have been proposed for constructing aggregation functions (see, e.g., [7, Chapters 5 and 6], or [30, Chapter 6]) as well as for building implication functions [4,5]. Among the latter, the pioneering and still very popular methods are those allowing to build implication functions from aggregation and negation functions by translating different classical logical

* Corresponding author. Tel.: +34 91 488 70 79.
*E-mail addresses:* ana.pradera@urjc.es (A. Pradera), gleb.beliakov@deakin.edu.au (G. Beliakov), bustince@unavarra.es (H. Bustince), bernard.debaets@ugent.be (B. De Baets).

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formulae to the fuzzy context. The well-established classes of strong implication functions ((S,N)-implications), residuated implication functions (R-implications), Quantum Logic implication functions (QL-implications) or Dishkant implication functions (D-implications, the contrapositions of QL-implications) are obtained in this way. Initially, the logical connectives $\wedge$ and $\vee$ were replaced with just two specific classes of aggregation functions, the well-known triangular norms and triangular conorms [2,34], providing interesting families of implication functions that have been widely studied (see [5] for appropriate references). However, it is clear that triangular norms and conorms are not the only aggregation functions that are able to model conjunctions and disjunctions, and hence many authors have considered the use of alternatives: see, e.g., [16,36,40,41,52–54,59] for works dealing with implication functions derived from uninorms or semi-uninorms, [35,58,60] for implication functions built from pseudo-t-norms, [17,24,62] for implication functions built from (dual) copulas, quasi-copulas and semi-copulas, [1,15] for implication functions generated from (dual) representable aggregation functions, [50] for implication functions generated from TS-functions and [19,20] for implication functions built from grouping/overlap functions.

A general approach to the construction of implication functions by means of other fuzzy connectives, and vice versa, was first proposed by Fodor in [25,26,28] after the work of Dubois and Prade [23]. A functional equation connecting implication functions and conjunctions was solved, and a new class of functions, the so-called weak triangular norms, was investigated. These results were deepened by Demirli and De Baets in [18], who dealt with a large class of binary operators and the logical equivalences $p \to q \equiv \neg(p \wedge \neg q)$ and $p \to q \equiv \bigvee\{t : p \wedge t \leq q\}$, which correspond to strong and residuated implication functions, respectively. These papers have not been as widely disseminated within the fuzzy set community as they should have been. In addition, on some aspects they are out of date, since the aggregation and implication fields have made important advances in the last few years.

A recent paper with general results on the construction of implication functions by means of aggregation functions revisits the subject [48]. It mainly reviews the case of residuated implication functions, although strong implication functions obtained via the material implication are also mentioned. The present paper follows the above direction reconsidering the equivalence $p \to q \equiv \neg p \vee q$ within the fuzzy framework.

We recall how this equivalence can be used not only to build implication functions from aggregation functions, but also in the opposite direction. We re-examine and systematize the relationships between the properties of implication functions and those of aggregation functions, illustrating them with several examples. Since the schemes $\neg p \vee q$ and $\neg(p \wedge \neg q)$ are built upon dual connectives, most of the results given in this paper are derived from those in [18,25,26].

The paper is organized as follows. Section 2 recalls the main issues regarding negation functions, aggregation functions and implication functions that are needed later on. Section 3 deals with some additional properties of aggregation functions that become relevant when studying their relationships with implication functions, focusing on the class of disjunctors. Sections 4 and 5 review the construction of implication functions from disjunctors, and vice versa, by means of the material implication, underscoring the relationships between their properties and providing some non-standard examples.

## 2. Basic definitions

This section encompasses the definitions and basic properties of negation, aggregation and implication functions.

### 2.1. Negation functions

**Definition 1** ([2,34,38]). A *negation function* is a decreasing function $N: [0,1] \to [0,1]$ verifying the boundary conditions $N(0) = 1$ and $N(1) = 0$. Strictly decreasing and continuous negation functions are known as *strict negations*, whereas involutive negation functions (i.e., those verifying $N(N(x)) = x$ for all $x \in [0,1]$) are known as *strong negations* (and constitute a subclass of strict negations).

**Example 2.** Some popular negation functions are described below (see, e.g., [34]).

- $N(x) = 1 - x$, which is a strong negation known as the classical or *standard negation*.
- $N(x) = 1 - x^2$, which is strict but not strong.
- The *smallest* and the *greatest* negation functions, given, respectively, by

$$N_{\perp}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and } N_{\top}(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{otherwise}. \end{cases}$$

These two negation functions belong to the class of *threshold negations*, those verifying $Ran(N) = \{0, 1\}$.

### 2.2. Aggregation functions

Although aggregation functions are defined for inputs of any cardinality, in this work we will only deal with *bivariate* aggregation functions.

**Definition 3.** A *(bivariate) aggregation function* is an increasing function $A: [0,1]^2 \to [0,1]$ verifying the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$.

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