Information Sciences 329 (2016) 434–446

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

On sets of graded attribute implications with witnessed non-redundancy

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ARTICLE INFO

Article history:
Received 21 May 2015
Revised 7 September 2015
Accepted 25 September 2015
Available online 3 October 2015

Keywords:
Redundancy
Attribute implication
Residuated lattice
Fuzzy logic
Pseudo intent

ABSTRACT

We study properties of particular non-redundant sets of if-then rules describing dependencies between graded attributes. The semantics of the considered rules is parameterized by linguistic hedges. In the setting of general idempotent truth-stressing hedges, we introduce notions of saturation and witnessed non-redundancy of sets of graded attribute implications. Furthermore, we show that bases of graded attribute implications given by systems of pseudo-intents correspond to non-redundant sets of graded attribute implications with saturated consequents where the non-redundancy is witnessed by antecedents of the contained graded attribute implications. For the special case of graded attribute implications parameterized by globalization, we introduce an algorithm which transforms any complete set of if-then rules into a base given by pseudo-intents. Experimental evaluation is provided to compare the method of obtaining bases with earlier graph-based approaches.

1. Introduction

In this paper, we introduce the notion of a witnessed non-redundancy of sets of graded attribute implications, study its properties and its relationship to the notion of a general system of pseudo-intents which has been introduced earlier [5]. The graded attribute implications (also known as fuzzy attribute implications) are if-then rules which generalize the ordinary attribute implications which appear in formal concept analysis [30]. The graded attribute implications are more general formulas than the classic attribute implications in that they allow to express attribute dependencies to degrees. For instance, a rule

$$\{{}^{0.9}/\mathit{good\ neighborhood}, {}^{1}/\mathit{large}\} \Rightarrow \{{}^{0.98}/\mathit{expensive}\} \qquad (1)$$

saying that if an object (e.g., a house for sale) is located in a good neighborhood and is large, then it is expensive, may be seen as a typical example of a graded attribute implication. In this example, the values 0.9, 1, and 0.98 (taken from the real unit interval) express lower bounds (or thresholds) of truth degrees to which we consider the attributes valid in data. Therefore, a finer reading of the rule is: "if a house is located in a good neighborhood *at least to degree* 0.9 and is large *at least to degree* 1, then it is expensive *at least to degree* 0.98". In formal concept analysis (FCA) of graded object-attribute data and in particular in the approach to FCA with linguistic hedges [17], graded attribute implications play an analogous role as the classic attribute implications in the ordinary FCA.

In FCA, one typically wants to find a small representative set of attribute implications which conveys the information about all attribute implications which hold in a given formal context. Equivalently, one wishes to find a small set of attribute implications

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http://dx.doi.org/10.1016/j.ins.2015.09.044
0020-0255/

whose models are exactly all concept intents of the data. Guigues–Duquenne bases [34] which are determined by pseudo-intents of formal concepts are examples of such sets which are in addition minimal in terms of the number of contained formulas, cf. also [29]. In FCA with graded attributes, a general notion of a system of pseudo-intents has been proposed and studied, see [18] for a survey. Unlike the classic case, general systems of pseudo-intents are not unique and may not ensure minimality of the corresponding base. Also, such systems may not exist when the structure of degrees is infinite and their existence for general finite scales is an open problem. From the computational point of view, graph-theoretic methods for obtaining general systems of pseudo-intents are proposed but they are limited only to small data sets. Therefore, further investigation is needed and this paper makes a contribution to this area.

In this paper, we show that bases of graded attribute implications given by general systems of pseudo-intents correspond to non-redundant sets of graded attribute implications with saturated consequents where the non-redundancy of each formula in the set is witnessed by its own antecedent. Both the notions of *saturation* and *witnessed non-redundancy* are introduced in Section 4. Furthermore, we introduce a constructive method for transformation of any set of graded attribute implications (which is complete in a given data) to a non-redundant base with witnessed non-redundancy from which a system of pseudo-intents can be derived. In practice, this means that we can avoid the graph-based method and compute systems of pseudo-intents by an alternative and much faster approach. We prove that the proposed procedure works if we consider globalization [51] as a parameter of the interpretation of graded attribute implications. For linguistic hedges [17,26,37] other than the globalization, which serve as parameters of the interpretation of graded attribute implications, the procedure may not produce the desired base but as our experimental observations show, it seems to have a high success rate.

The results contained in our paper fall in the category of results on bases of if-then rules generated from data [3,19,43] which develop ideas of the seminal paper [34]. Although we work with graded if-then rules with semantics defined using complete residuated lattices as structures of degrees and parameterized by linguistic hedges, our approach is general and we anticipate it can be adopted in recently developed approaches such as [40,45].

Our paper is organized as follows. In Section 2, we present the basic notions of residuated structures of truth degrees and graded attribute implications. In Section 3, we present a background and a survey of existing results on non-redundant bases of graded attribute implications and give further motivation for our work. Section 4 contains the new results. Finally, Section 5 shows experimental observations on efficiency on computing sets of graded attribute implications with witnessed non-redundancy and presents open problems.

2. Preliminaries

In this section, we present the basic notions of structures of truth degrees and graded attribute implications. Whenever possible, we keep the same notation as in [4] for general residuated structures and [18] for graded attribute implications.

We utilize complete residuated lattices as structures of truth degrees. For our development, these structures represent a reasonable generalization of the most common structures of degrees defined on the real unit interval using left-continuous triangular norms [25,38]. Recall that a complete residuated lattice [4,28] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (i.e., a lattice where infima and suprema exist for arbitrary subsets of L), $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and 1 is neutral with respect to $\otimes$), and $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property: for all $a, b, c \in L$, we have that $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, where $\leq$ is the (complete lattice) order induced by $\mathbf{L}$ (i.e., $a \leq b$ iff $a = a \wedge b$ iff $a \vee b = b$ iff $a \rightarrow b = 1$). We interpret $\otimes$ and $\rightarrow$ as it is usual in mathematical fuzzy logics [20,32,33,36] and their applications [39]: $\otimes$ is a truth function of "fuzzy conjunction" and $\rightarrow$ is a truth function of "fuzzy implication", cf. also [21,22] for surveys of results on fuzzy logics in the narrow sense.

In the paper, we use illustrative examples based on finite (and thus complete) residuated lattices defined on equidistant subchains of the real unit interval. That is, we consider $L = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ for some natural n and use the natural ordering of rational numbers, i.e., $\wedge$ and $\vee$ coincide with the operations of minimum and maximum, respectively. If $\otimes$ coincides with $\wedge$, we call the resulting $\mathbf{L}$ a finite Gödel chain in which case we have $a \rightarrow b = 1$ iff $a \leq b$ and $a \rightarrow b = b$ otherwise. If $\otimes$ and $\rightarrow$ are given by

$$\tfrac{i}{n} \otimes \tfrac{j}{n} = \max\left\{0, \tfrac{i}{n} + \tfrac{j}{n} - 1\right\}, \tag{2}$$

$$\tfrac{i}{n} \rightarrow \tfrac{j}{n} = \min\left\{1, 1 - \tfrac{i}{n} + \tfrac{j}{n}\right\}, \tag{3}$$

we call the resulting $\mathbf{L}$ a finite Łukasiewicz chain. More general finite residuated lattices on equidistant subchains of the real unit interval may be considered but in our examples we utilize only these two basic structures, cf. [23,38].

In addition to $\otimes$ and $\rightarrow$ which may be seen as generalizations of truth function of the classic logical connectives "conjunction" and "implication", we make use of linguistic hedges [55–58] which do not have nontrivial counterparts in classic logics. In particular, we utilize idempotent truth-stressing (i.e., truth intensifying) linguistic hedges (shortly, hedges), which are considered as maps ${}^*\colon L \rightarrow L$ such that $1^* = 1$, $a^* \leq a$, $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, and $a^* \leq a^{**}$ for all $a, b \in L$. Using similar arguments as in [37], such maps may be seen as truth functions of logical connectives "very true", cf. also [26] and [22] for recent results on hedges. In our setting, the hedges are idempotent, meaning that $a^{**} = a^*$. As a consequence, each hedge is a particular interior operator on $\mathbf{L}$ and from the point of view of their interpretation, "very very true" represents the same emphasis on truth as "very true". This property is usually not considered as a basic property of hedges [55–58] and we keep it for technical reasons which originated in their initial use in FCA [10,17].