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Deresiduums of implications on a complete lattice

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ABSTRACT

In this paper, we first discuss some properties of residuum and deresiduum of a binary operation on a complete lattice. Then, we investigate the relations between residuum and deresiduum. Finally, we demonstrate how each individual logical character of a right infinitely $\wedge$-distributive implication, such as the law of importation, the weak exchangeability principle, the exchange principle, the contrapositive symmetry and the contraction law, can be translated into a property of its deresiduum. Moreover, we give some conditions under which deresiduum of a right infinitely $\wedge$-distributive implication is, respectively, a left (right) uninorm, pseudo-uninorm and uninorm, and show that right infinitely $\wedge$-distributive implications, which satisfy the weak exchangeability principle and the contrapositive symmetry, can be presented by commutative conjunctors and strong negations.

1. Introduction

Implications are an essential tool in fuzzy control and approximate reasoning, as well as in many fields where these theories apply. This is because they are used not only to model fuzzy conditionals, but also to make inferences in fuzzy rule-based system (see for instance [22,27,32]) through the modus ponens and modus tollens. Moreover, they are useful not only in fuzzy control and approximate reasoning, but also in many other fields like fuzzy relation equations [32], computing with words [32,46], fuzzy DI-subsethood measures and image processing [6,7,24], fuzzy neural networks [39–41], fuzzy mathematical morphology [13,24,41] and data mining [45]. Residual implications on [0, 1] have been studied by many authors (see [1,9,14,32,42]) and extended to a complete lattice (see [12,21,34,38]).

Deresiduation is introduced by Durante et al. [14]. On the one hand, some authors used deresiduum to construct special conjunctions such as semi-copula, pseudo t-norm and t-norm (see [14,26] for detail) and semi-uninorm [29]. On the other hand, residuum and deresiduum of a binary operation are process relating conjunction and implication in the semantic of logics, either classical or not. From the mathematics point of view they arise from isotonic Galois connection, also called adjunction, between ordered sets and characterize residuated structures, in particular residuated lattices (see [19] for details). The adjunction relationship is important in lots of application fields. For example, adjunction relationship forms the backbone of learning strategy of fuzzy morphological associative memories (see [40,41] for details).

In this paper, motivated by these works on [14,26,29], we further study the deresiduums of implications on a complete lattice. In Section 2, we briefly recall some concepts which will be used in the paper. In Section 3, we investigate some properties of residuum and deresiduum and discuss some relations between residuum and deresiduum. In Section 4, we demonstrate how each individual logical character of a right infinitely $\wedge$-distributive implication, such as the law of importation, the neutral

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principle, the weak exchangeability principle, the exchange principle, the contrapositive symmetry and the contraction law, can be translated into a property of its deresiduum, give some conditions under which deresiduum of an implication is, respectively, a left (right) uninorm, pseudo-uninorm and uninorm, and show that right infinitely $\wedge$-distributive implications, which satisfy the weak exchangeability principle and the contrapositive symmetry, can be presented by commutative conjunctors and strong negations.

The knowledge about lattices required in this paper can be found in [5].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Preliminaries

In this section, we briefly recall some concepts and results which will be used in the sequel.

Definition 2.1. A binary operation A on L is said to

(1) have a left neutral element e, if $A(e,x)=x$ holds for any $x\in L$;
(2) have a right neutral element e, if $A(x,e)=x$ holds for any $x\in L$;
(3) have a neutral element e, if $A(x,e)=A(e,x)=x$ holds for any $x\in L$;
(4) be commutative, if $A(x,y)=A(y,x)$ holds for any $x,y\in L$;
(5) be associative, if $A(x,A(y,z))=A(A(x,y),z)$ holds for any $x,y,z\in L$.

Definition 2.2. (Ma and Wu [31], Wang and Yu [44]). A mapping $N: L\to L$ is called a strong negation if
$(N1)$ $N(N(a))=a\ \forall a\in L$,
$(N2)$ $a\le b,\ a,b\in L\Rightarrow N(b)\le N(a)$.
Clearly, $N(0)=1$ and $N(1)=0$ for any strong negation on L.

Theorem 2.1. *(Wang and Yu [44]). Let $a_j\in L(j\in J)$. If N is a strong negation on L, then*

$$N\left(\bigvee_{j\in J}a_j\right)=\bigwedge_{j\in J}N(a_j), \tag{2.1}$$

$$N\left(\bigwedge_{j\in J}a_j\right)=\bigvee_{j\in J}N(a_j). \tag{2.2}$$

Definition 2.3. (De Baets [8]). Consider a strong negation N on L. The N-dual operation of a binary operation A on L is the binary operation A_N on L defined by

$$A_N(a,b)=N(A(N(a),N(b)))\quad\forall a,b\in L. \tag{2.3}$$

Note that $(A_N)_N=A$ for any binary operation A on L.

Definition 2.4. (De Baets [8], Durante et al. [14]). A conjunctor is a binary operation $C: L\times L\to L$ that is increasing in each variable and such that its restriction to $\{0,1\}\times\{0,1\}$ is a boolean conjunction, i.e., it satisfies $C(0,1)=C(1,0)=0$ and $C(1,1)=1$.

Definition 2.5. (Wang and Fang [42,43]). A binary operation U on L is called a left (right) uninorm if it is increasing, associative and has a left (right) element $e_L\in L$ ($e_R\in L$).

If a left (right) uninorm U with a left (right) element e_L (e_R) has a right (left) element e_R (e_L), then $e_L=U(e_L,e_R)=e_R$. Let $e=e_L=e_R$. Here, we call U pseudo-uninorm on L and e the neutral element of U. Moreover, when U is commutative, we call U uninorm on L.

Semi-uninorms introduced by Liu [29] are non-associative pseudo-uninorms.

Definition 2.6. (Wang and Fang [42,43]). A binary operation A is called left (right) infinitely $\vee$-distributive if, for all index sets J, the following statements are valid:

$$A\left(\bigvee_{j\in J}x_j,y\right)=\bigvee_{j\in J}A(x_j,y)\left(A\left(x,\bigvee_{j\in J}y_j\right)=\bigvee_{j\in J}A(x,y_j)\right)\quad\forall x,y,x_j,y_j\in L;$$

left (right) infinitely $\wedge$-distributive if, for all index sets J, the following statements are valid:

$$A\left(\bigwedge_{j\in J}x_j,y\right)=\bigwedge_{j\in J}A(x_j,y)\left(A\left(x,\bigwedge_{j\in J}y_j\right)=\bigwedge_{j\in J}A(x,y_j)\right)\quad\forall x,y,x_j,y_j\in L.$$

If a binary operation A is left infinitely $\vee$-distributive ($\wedge$-distributive) and also right infinitely $\vee$-distributive ($\wedge$-distributive), then A is said to be infinitely $\vee$-distributive ($\wedge$-distributive).

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