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Convergences and topology via sequences of multifunctions

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ABSTRACT

In this paper we define different types of convergences for sequences of multifunctions and establish relationships among them. Using the convergence in fuzzy measure we obtain a topological structure on a space of multifunctions.

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1. Introduction

Beginning with Choquet [6], Zadeh [37] and Sugeno [33], the fuzzy theory proved its importance and utility in numerous applications (e.g. [1,20–23,34]).

In set-valued analysis, the theory of measurable multifunctions has various applications in statistics, mathematical economics, fixed point problems, theory of control, optimization, information systems, modeling birth-and-growth processes, game theory (e.g. [2,4,12–14,26,28,29]). The theory of multifunctions or set-valued functions (i.e. functions with values in a family of sets) has been studied by many authors (e.g. [3,7–11,15–18,27,30–32]). For (single-valued) real or vector functions $f : T \to X$, one usually deals with the absolute value or the norm. Passing to the theory of multifunctions (i.e. functions with values in $\mathcal{P}_0(X)$, where $\mathcal{P}_0(X)$ is the family of all nonempty subsets of X) arises difficult aspects since the definitions of $\mathcal{P}_0(X)$-valued cannot be reduced to that of X-valued. This happens because if X is a normed space, then $P_0(X)$ is not a linear space: in fact, it is not a group with respect to the usually addition "+" defined by

$$M + N = \{x + y | x \in M, y \in N\}, \text{for every } M, N \in P_0(X).$$

So, special considerations have to be introduced. For this reason, in the present paper different set-valued settings will be made using a set-norm on $\mathcal{P}_0(X)$ (see Definition 1).

In this paper we define different types of convergences for sequences of multifunctions and establish some relationships among them. Then we introduce a topology on a space of multifunctions via neighborhood systems defined by convergence in fuzzy measure for sequences of multifunctions.

The paper is organized as follows: in Section 2 we give some preliminaries. In Section 3 we define different types of convergences for sequences of multifunctions and establish some relationships among them. In Section 4 we introduce a topology on a space of multifunctions, offering a work frame in studying this important field of set-valued analysis. The

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topology is defined using convergence in a fuzzy measure for sequences of multifunctions. Section 5 presents some applications and Section 6 is for the conclusion.

2. Preliminaries

In this section some useful definitions and remarks are presented.

The set of all real numbers is denoted by $\mathbb{R}$. We denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where $\mathbb{N}$ is the set of all nonnegative integers.

Let X be a linear space and $\mathcal{P}_0(X)$ the family of all nonempty subsets of X. If "$\leqslant$" is an order relation on $\mathcal{P}_0(X)$, then we denote this by $(\mathcal{P}_0(X), \leqslant)$. For example, the usual set inclusion "$\subseteq$" is an order relation on $\mathcal{P}_0(X)$, that will be denoted by $(\mathcal{P}_0(X), \subseteq)$. For every $E, F \in \mathcal{P}_0(X)$ and $\alpha \in \mathbb{R}$, denote $E + F = \{x + y | x \in E, y \in F\}$ and $\alpha E = \{\alpha x | x \in E\}$.

Definition 1. ([8]) Let X be a linear space. A function $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is called *a set-norm* on $\mathcal{P}_0(X)$ if:

(i) $|E| = 0 \Longleftrightarrow E = \{0\}$, for $E \in \mathcal{P}_0(X)$;
(ii) $|\alpha E| = |\alpha| \cdot |E|, \forall \alpha \in \mathbb{R}, \forall E \in \mathcal{P}_0(X)$ (with the convention $0(+\infty) = 0$);
(iii) $|E + F| \leqslant |E| + |F|, \forall E, F \in \mathcal{P}_0(X)$.

Definition 2. A set-norm $|\cdot|$ on $(\mathcal{P}_0(X), \leqslant)$ is called *monotone* if for every sets $E, F \in \mathcal{P}_0(X), E \leqslant F \Rightarrow |E| \leqslant |F|$.

Remark 3.

I. If X is a linear space and $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is a set-norm on $\mathcal{P}_0(X)$ such that $|\{x\}| < +\infty, \forall x \in X$ then the function defined by $\|x\| = |\{x\}|, \forall x \in X$, is a norm on X.
II. If $(X, \|\cdot\|)$ is a normed space, then the function $|E|_s = \sup_{x \in E} \|x\|$ is a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$, called the supremum set-norm on $\mathcal{P}_0(X)$.
III. Let X be a linear space and $\|\cdot\|_1$ a norm on X. Then, according to II, we obtain a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$, defined by $|E|_s = \sup_{x \in E} \|x\|_1$ for every $E \in \mathcal{P}_0(X)$. By I, we obtain $\|x\|_2 = |\{x\}|_s$, for every $x \in X$. It results that $\|x\|_2 = \|x\|_1$ for every $x \in X$.
IV. Let X be a linear space and $|\cdot|_1$ a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$ so that $|\{x\}|_1 < +\infty$, for every $x \in X$. By I, the function defined by $\|x\| = |\{x\}|_1$ is a norm on X. According to II, we obtain another set-norm $|E|_2 = \sup_{x \in E} \|x\| = |E|_s, \forall E \in \mathcal{P}_0(X)$. Then $|E|_2 \leqslant |E|_1$, for every $E \in \mathcal{P}_0(X)$ (the inequality may be strict as we can see in Example 4). Indeed, let $E \in \mathcal{P}_0(X)$. Since the set-norm is monotone, we have $\|x\| = |\{x\}|_1 \leqslant |E|_1$, for every $x \in E$. Now, we obtain $|E|_2 = |E|_s = \sup_{x \in E} \|x\| \leqslant |E|_1$.

Moreover, if $(X, \|\cdot\|)$ is a normed space and $|E|_1 = \sup_{x \in E} \|x\|$, then $|E|_2 = |E|_1, \forall E \in \mathcal{P}_0(X)$.

Example 4. Let $X = \mathbb{R}$ and for every $E \in \mathcal{P}_0(\mathbb{R})$, let

$$|E|_1 = \begin{cases} \max_{x \in E} |x|, & \text{if } E \text{ is finite} \\ +\infty, & \text{if } E \text{ is not finite.} \end{cases}$$

Then $|\cdot|_1$ is a monotone set-norm on $(\mathcal{P}_0(\mathbb{R}), \subseteq)$. By Remark 3-IV, we obtain another monotone set-norm on $(\mathcal{P}_0(\mathbb{R}), \subseteq)$, that is $|E|_2 = \sup_{x \in E} |x|$, for every $E \in \mathcal{P}_0(\mathbb{R})$. Let $E = (-5, 4]$. Then $|E|_2 = 5 < +\infty = |E|_1$.

In the sequel, let T be a nonempty set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C} \subseteq \mathcal{P}(T)$ a ring of subsets of T (i.e. $\mathcal{C} \neq \emptyset, A \cup B \in \mathcal{C}$ and $A \setminus B \in \mathcal{C}$ for every $A, B \in \mathcal{C}$).

Definition 5. A set function $\mu : \mathcal{C} \to [0, +\infty]$ is called:

(i) *monotone* if $\mu(A) \leqslant \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.
(ii) *a fuzzy measure* if μ is monotone and $\mu(\emptyset) = 0$.
(iii) *strongly order-continuous* (shortly *strongly o-continuous*) if $\lim_{n \to \infty} \mu(A_n) = 0$, for every $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ with $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}, \bigcap_{n=0}^{\infty} A_n = A, A \in \mathcal{C}$ and $\mu(A) = 0$ (denoted by $A_n \searrow A$).
(iv) *autocontinuous from above* if $\lim_{n \to \infty} \mu(A \cup B_n) = \mu(A)$, for every $A \in \mathcal{C}$ and $(B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ with $\lim_{n \to \infty} \mu(B_n) = 0$.
(v) *autocontinuous from below* if $\lim_{n \to \infty} \mu(A \setminus B_n) = \mu(A)$ for every $A \in \mathcal{C}$ and $(B_n) \subset \mathcal{C}$ with $B_n \subseteq A$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} \mu(B_n) = 0$.

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