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Algebraic structures in the vicinity of pre-rough algebra and their logics

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ABSTRACT

In this paper a cluster of algebraic structures weaker than pre-rough algebra has been studied. Their properties and interrelations are investigated. Logics, both Hilbert type axiomatization and sequent calculi for these algebras are presented. These algebraic structures may be considered as abstract generalizations of rough set algebra. In particular some of the algebras are observed to be quite rich in structure and hence bear the potentiality of generating concrete rough set models for specific usages.

1. Introduction

Since the inception of rough set theory in 1982 [11], various abstract algebraic structures have emerged depending upon various ways of defining rough sets. An account of many of these structures may be found in [4]. One such structure is called pre-rough algebra which was first defined in [2]. In the same paper [2] a more restricted structure called rough algebra was also defined and a representation theorem akin to Stone representation theorem for Boolean algebras was established. Subsequently it was observed [3] that any pre-rough algebra is isomorphic to some pre-rough algebra of classical (Pawlakian) rough sets in some approximation space when the operations are restricted only to the finitary ones. Predecessors of pre-rough and rough algebras are quasi Boolean algebra **(qBa)** (also called De Morgan lattice) and topological quasi Boolean algebras. Although the notion of topological Boolean algebra **(tqBa)** was already present in the extant literature [12], topological quasi Boolean algebra was first defined in [1]. In one of the issues of Information Sciences [6] an algebraic structure called MDS5 algebra has been extensively studied in the general perspective of abstraction of rough sets. This algebra is very close to pre-rough algebra and if distributivity is added it occupies a position between topological quasi Boolean algebra and pre-rough algebra. We call this algebra MD′S5. Another important algebraic structure called Tetravalent Modal Algebra **(TMA)** was studied from a completely different motivation in [10,7]. It is observed that TMA lies between topological quasi Boolean algebra and pre-rough algebra in the sense that it is stronger than topological quasi Boolean algebra and weaker than pre-rough algebra. Following subsequent terminology which has been observed to be more appropriate, we shall split the

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original notion of topological quasi Boolean algebra into two notions viz. topological quasi Boolean algebra and topological quasi Boolean algebra 5 **(tqBa5)** which is topological quasi Boolean algebra, with one more axiom (the algebraic counterpart of the modal logic axiom S_5 [8]). Fig. 3 of Section 3 depicts the relationship between the algebras topological quasi Boolean algebra, topological quasi Boolean algebra 5, MD′S5, TMA and pre-rough algebra.

In pre-rough algebra a binary operator 'arrow' ($\Rightarrow$) was defined in terms of the other operators which has the natural interpretation in the context of classical rough set corresponding to the notion of rough inclusion [1]. This operator has been instrumental to the development of pre-rough and rough logics [2] in obtaining the logical connective implication ($\rightarrow$) which is crucial to the definition of a logic. It is known that the rule modus ponens (MP) i.e., to infer β from the set $\{\alpha, \alpha \rightarrow \beta\}, \alpha, \beta$ being wffs, is the basic rule of inference in almost all the important logics. For being a useful interpretation of $\rightarrow$ the algebraic operator $\Rightarrow$ has to have the following property: $a \leqslant b$ if and only if $a \Rightarrow b = 1$, 1 being the top element of the lattice in which an interpretation of the logic is sought for. That such an arrow operator is required and was not apparently available in topological quasi Boolean algebra or topological quasi Boolean algebra 5 had been recorded in [2]. Unfortunately, no such arrow exists in topological quasi Boolean algebra 5 (and hence in topological quasi Boolean algebra) as was established subsequently in [5]. For this reason it was not possible to formulate a Hilbert system type logic for the algebraic structures topological quasi Boolean algebra or topological quasi Boolean algebra 5, although a sound and complete sequent calculus was proposed in [14]. Another kind of implication operator is defined in [16] which may also be of interest to researchers on rough sets.

There was another unresolved problem in [13]. That is the question of independence of the axioms of pre-rough algebra particularly of the three Axioms 7–9 (see Definition 2.6) viz.

$\sim Ia \vee Ia = 1$, for all $a \in A$,

$I(a \vee b) = Ia \vee Ib$, for all $a, b \in A$ and

$Ca \leqslant Cb, Ia \leqslant Ib$ imply $a \leqslant b$, for all $a, b \in A$.

In a subsequent work [14], it was observed that axioms $\sim Ia \vee Ia = 1$, for all $a \in A$ and $Ca \leqslant Cb, Ia \leqslant Ib$ imply $a \leqslant b$, for all $a, b \in A$ are independent of the others while the question of independence of axiom $I(a \vee b) = Ia \vee Ib$, for all $a, b \in A$, remained open. In [6] one of the axiom viz. $IIa = Ia$, for all $a \in A$, was proved to follow from the others. But a more extensive study of independence has not been carried out.

The present work is a logic-algebraic study of a number of algebraic structures in the vicinity of pre-rough algebra. In particular we investigate structures in which an arrow operator $\Rightarrow$ of the type of pre-rough algebra would be available so that Hilbert type logics could be developed for these structures. To our amazement it is discovered that there exists a cluster of such structures just before pre-rough algebra. These algebras were as if hidden near (in between) the two main structures topological quasi Boolean algebra 5 and pre-rough algebra. So, in a sense, the question of availability of the arrow has been answered, though not completely but to a large extent. Besides, the number of axioms for making a pre-rough algebra has been considerably reduced and the question of their independence is decided. The automated theorem prover (Prover9)[9] has been used extensively in dealing with this issue. Taking help of Prover9, proofs have been constructed afterwards which are presented in the appendices.

Another dimension, a new one, has been added to the study of this cluster of algebras. In the algebraic studies of rough sets there is a notion which is the counterpart of the property which characterizes the partition space in topology [15] viz. a set is open if and only if it is closed. The definable sets of an approximation space in Pawlakian rough set theory possess the property. This notion has been studied in depth in abstract rough set literature [6]. We have investigated which structures defined in this paper possess this property. Availability of this property in algebra will foster construction of lower and upper approximations of an element in an abstract rough set setting.

This paper has been organized as below.

In Section 2, a modified and reduced set of axioms of pre-rough algebra is presented. Section 3 defines the bunch of algebraic structures weaker than pre-rough algebra, investigates their properties and inter-relations. Section 4 deals with the logic-systems – Hilbert type axiomatic systems and sequent calculi – for these algebraic structures. Section 5 contains some concluding remarks. Appendices containing the proofs developed with the help of Prover9 has been added at the end.

2. Reduction of the axioms of pre-rough algebra

The pair $\langle X, R\rangle$ where X is a non-empty set and R is an equivalence relation in it, is called an approximation space.

For any $A \subseteq X$, the lower and upper approximations are defined by

$$\underline{A} = \{x | [x]_R \subseteq A\}$$
$$\overline{A} = \{x | [x]_R \cap A \neq \phi\}.$$

Sets A and B are called roughly equal if and only if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$. A relation $\approx$ in $\mathcal{P}(x)$ defined by $A \approx B$ if and only if A and B are roughly equal, is an equivalence relation. An equivalence class $[\bullet]_\approx$ of $\mathcal{P}(x)/_\approx$ is called a rough set (Pawlakian or classical). There are other definitions too [2]. Operations $\sqcap, \sqcup, \neg, \mathsf{L}$ are defined in $\mathcal{P}(x)/_\approx$ by