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# Conditional connectivity of recursive interconnection networks respect to embedding restriction ☆

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ABSTRACT

Large-scale multiprocessor systems always take some recursive interconnection networks as underlying topologies. Let $G_n$ be an $n$-dimensional recursive interconnection network. The $m$-embedding-restricted connectivity $\zeta_m(G_n)$ (resp. the $m$-embedding-restricted edge connectivity $\eta_m(G_n)$) of $G_n$ is the cardinality of a minimum subset of nodes (resp. edges), if any, whose deletion disconnects $G_n$ and each node of the remaining components lies in an undamaged $m$-dimensional sub-network $G_m$. In this paper, we present some relationships between the proposed indices and other conditional connectivity indices in general recursive interconnection networks. We give some bounds on these two indices in $k$-ary $n$-cubes and bubble-sort networks. In addition, we determine these two indices in $k$-ary $n$-cubes and bubble-sort networks in some cases.



## 1. Introduction and preliminaries

In this paper, we follow [6] for the graph-theoretical terminology and notation not defined here.

In a multiprocessor system, processers are connected according to some interconnection network which is usually represented by a graph $G = (V, E)$, where each node in $V$ corresponds to a processor, and each edge in $E$ corresponds to a communication link. The properties of an interconnection network determine the corresponding system's performance. Reliability is an important property to be considered when selecting or designing an interconnection network for a multiprocessor system. The connectivity $\kappa(G)$ (resp. edge connectivity $\lambda(G)$) of a connected graph $G$ is the cardinality of a minimum subset of nodes (resp. edges), if any, whose deletion disconnects $G$. Connectivity and edge connectivity are two important indices to evaluate the reliability of a network. The two parameters, however, have an obvious deficiency, that is, they tacitly assume that all nodes adjacent to (or all edges incident with) a node can potentially fail simultaneously, which is almost impossible in a real multiprocessor system. To compensate for this shortcoming, Harary [13] introduced the concept of conditional connectivity by imposing some conditions or restrictions on the remaining components of the graph after deleting some nodes or edges. Following this trend, restricted connectivity and restricted edge connectivity were proposed in [7,11,12,14]; extraconnectivity was proposed and studied in [12,29,30]; and $m$-restricted connectivity, $m$-restricted edge connectivity, $R^m$-restricted connectivity and $R^m$-edge-connectivity were addressed and explored in [5,8,9,18,19,23,24,26,28,32].

An *m-restricted cut* (resp. *m-restricted edge cut*) of a graph $G$ is a set of nodes (resp. edges), if any, whose deletion disconnects $G$ and every remaining component has at least order $m$. The cardinality of a minimum $m$-restricted cut (resp.

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$m$-restricted edge cut) is the *$m$-restricted connectivity* (resp. *$m$-restricted edge connectivity*) of $G$ and is denoted by $\kappa_m(G)$ (resp. $\lambda_m(G)$). 2-Restricted connectivity (resp. 2-Restricted edge connectivity) is also called *restricted connectivity* (resp. *restricted edge connectivity*) and is often denoted by $\kappa'(G)$ (resp. $\lambda'(G)$). An *$R^m$-cut* (resp. *$R^m$-edge-cut*) of a graph $G$ is a set of nodes (resp. edges), if any, whose deletion disconnects $G$ and each node of the remaining components has at least $m$ neighbors. The cardinality of a minimum $R^m$-cut (resp. $R^m$-edge-cut) is the *$R^m$-connectivity* (resp. *$R^m$-edge connectivity*) of $G$ and is denoted by $\kappa^m(G)$ (resp. $\lambda^m(G)$).

In a real multiprocessor system, the presence of node and/or edge failures will make the entire network faulty and maybe the network is not connected any more. In this scenario, people hope that every remaining component of the network has undamaged subnetworks (i.e., smaller networks with the same topological properties as the original one) so as to use the same routing algorithm or maintenance strategy as used in the original one. Under this consideration, we [31] introduced two indices to evaluate the connectivity of recursive interconnection networks. Define an *$m$-embedding-restricted cut* (resp. *$m$-embedding-restricted edge cut*) of a recursive interconnection network to be a set of nodes (resp. edges), if any, whose deletion disconnects $G$ and each node of the remaining components lies in an undamaged $m$-dimensional subnetwork. The cardinality of a minimum $m$-embedding-restricted cut (resp. $m$-embedding-restricted edge cut) is *the $m$-embedding-restricted connectivity* (resp. *$m$-embedding-restricted edge connectivity*) of $G$ and is denoted by $\zeta_m(G)$ (resp. $\eta_m(G)$). Then what are the exact values of the $m$-embedding-restricted connectivity and the $m$-embedding-restricted edge connectivity of some popular networks? In [31], we studied the two indices in the star network $S_n$. In this paper, we make a further research on these two indices in more recursive interconnection networks. For convenience, we simplify the terms $m$-embedding-restricted cut and $m$-embedding-restricted edge cut as *$m$-ER cut* and *$m$-ER edge cut*, respectively. Given an $n$-dimensional recursive interconnection networks $G_n$, for completeness, let $\zeta_m(G_n) = +\infty$ (resp. $\eta_m(G_n) = +\infty$) if $G_n$ has no $m$-ER cut (resp. $m$-ER edge cut).

The rest of the paper is organized as follows. In Sections 2, we prove some results on the two indices in general recursive interconnection networks. In Sections 3 and 4, we investigate the $m$-embedding-restricted connectivity and the $m$-embedding-restricted edge connectivity in $k$-ary $n$-cubes and bubble-sort networks. Conclusions and discussions are covered in Section 5.

## 2. General recursive interconnection networks

In this section, we will show some results on the $m$-embedding-restricted connectivity and the $m$-embedding-restricted edge connectivity in general recursive interconnection networks.

**Lemma 2.1.** *Let $n$ and $m$ be two integers with $n \geqslant 2$ and $1 \leqslant m \leqslant n-1$ and let $G_n$ be an $n$-dimensional recursive interconnection network. Then $\eta_m(G_n) \geqslant \lambda_{|V(G_m)|}(G_n)$.*

**Proof.** If $G_n$ has no $m$-ER edge cut, by the definition, $\eta_m(G_n) = +\infty$, and so the lemma is trivial. If $G_n$ has an $m$-ER edge cut, let $F_e \subset E(G_n)$ be a minimum $m$-ER edge cut of $G_n$. Then $\eta_m(G_n) = |F_e|$, $G_n - F_e$ is not connected and each node of $G_n - F_e$ lies in an $m$-dimensional subnetwork $G_m$ of $G_n$. Note that $G_m$ has $|V(G_m)|$ nodes. Therefore, each component of $G_n - F_e$ has at least $|V(G_m)|$ nodes. Thus, $|F_e|$ is a $|V(G_m)|$-restricted edge cut of $G_n$, which implies that $\lambda_{|V(G_m)|}(G_n) \leqslant |F_e|$. Combining this with $\eta_m(G_n) = |F_e|$, we have $\eta_m(G_n) \geqslant \lambda_{|V(G_m)|}(G_n)$. $\square$

**Lemma 2.2.** *Let $n$ be an integer, $G_n$ be an $n$-dimensional $d_n$-regular recursive interconnection network and let $m$ be an integer with $1 \leqslant m \leqslant n-1$. Then $\zeta_m(G_n) \geqslant \kappa^{d_m}(G_n)$.*

**Proof.** If $G_n$ has no $m$-ER cut, the lemma is trivial. If $G_n$ has an $m$-ER cut, let $F_v \subset V(G_n)$ be a minimum $m$-ER cut of $G_n$. Then $\zeta_m(G_n) = |F_v|$, $G_n - F_v$ is not connected and each node of $G_n - F_v$ lies in an $m$-dimensional subnetwork $G_m$ of $G_n$. Note that $G_m$ is $d_m$ regular. Therefore, each node of $G_n - F_v$ has at least $d_m$ fault-free neighbors. Thus, $|F_v|$ is an $R^{d_m}$-cut of $G_n$, which implies that $\kappa^{d_m}(G_n) \leqslant |F_v|$. Combining this with $\zeta_m(G_n) = |F_v|$, we have $\zeta_m(G_n) \geqslant \kappa^{d_m}(G_n)$. $\square$

**Lemma 2.3.** *Let $n$ be an integer, $G_n$ be an $n$-dimensional $d_n$-regular recursive interconnection network and let $m$ be an integer with $1 \leqslant m \leqslant n-1$. Then $\eta_m(G_n) \geqslant \lambda^{d_m}(G_n)$.*

**Proof.** If $G_n$ has no $m$-ER edge cut, the lemma is trivial. If $G_n$ has an $m$-ER edge cut, let $F_e \subset E(G_n)$ be a minimum $m$-ER edge cut of $G_n$. Then $\eta_m(G_n) = |F_e|$, $G_n - F_e$ is not connected and each node of $G_n - F_e$ lies in an $m$-dimensional subnetwork $G_m$ of $G_n$. Note that $G_m$ is $d_m$-regular. Therefore, each node of $G_n - F_e$ has at least $d_m$ neighbors. Thus, $|F_e|$ is an $R^{d_m}$-edge-cut of $G_n$, which implies that $\lambda^{d_m}(G_n) \leqslant |F_e|$. Combining this with $\eta_m(G_n) = |F_e|$, we have $\eta_m(G_n) \geqslant \lambda^{d_m}(G_n)$. $\square$

**Theorem 2.1.** *Let $n$ be an integer and $G_n$ be an $n$-dimensional regular recursive interconnection network. If there exists an integer $1 \leqslant m \leqslant n-1$ such that $G_m$ is a complete graph on two nodes, then $\zeta_m(G_n) = \kappa^1(G_n) = \kappa_2(G_n)$ and $\eta_m(G_n) = \lambda^1(G_n) = \lambda_2(G_n)$.*

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