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Triadic concept lattices in the framework of aggregation structures ☆

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ABSTRACT

Triadic concept analysis is a method of analysis of three-way tabular data describing objects by binary attributes they have under various conditions. It aims to extract a set of three-dimensional clusters, so-called triadic concepts, from the data. While the method was recently extended to handle the case of graded data, this extension is only one of a variety of possible generalizations, namely the one where the so-called antitone concept-forming operators are considered. The generalizations not covered by the mentioned approach include isotone concept-forming operators, one sided concepts, the use of truth stressing hedges and many others. Since all of the generalizations have their applications, for example in factor analysis of relational data, it is desirable to have a uniform mathematical foundations covering all of the cases. In this paper we provide such foundations. By using triadic aggregation structures as a scale of truth degrees we develop triadic concept analysis in a very general setting that covers a wide variety of its possible extensions in a uniform way.

1. Introduction

The triadic concept analysis (TCA) was introduced by Wille and Lehmann in [23] as an extension of (dyadic) formal concept analysis (FCA). While the formal concept analysis formalizes the notion of a concept as understood in Port-Royal logic, where a concept is considered to be a basic element of thought represented by its extension and intension, the development of triadic concept analysis was further motivated by the works of the philosopher Charles Peirce and his system of categories. Both FCA and TCA proved to be applicable in various domains, including dimensionality reduction (see e.g. [10,4]), information retrieval [28], or machine learning [27].

In our previous work [7], we studied a generalization of TCA to graded data (fuzzy setting). In the present paper, we pursue the generalization even further.

Antitone fuzzy Galois connections and concept lattices, the basic mathematical structures behind FCA, were in the setting of graded data studied e.g. in [3,29]. In [14,30] an alternative approach based on isotone fuzzy Galois connections was studied. It is well known that in the ordinary setting (that is data without grades), the antitone and isotone cases are mutually reducible due to the law of double negation. In a graded setting only one sided reducibility can be obtained [6]. Nevertheless, a unifying framework which covers both, the antitone and isotone case, was recently proposed in [1,5], see also [14]. For other relevant generalizations of FCA see [21,22,24–26].

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Inspired by this approach, we provide a unifying framework which enables to treat the antitone and isotone cases in TCA as its special instances. We provide mathematical foundations of the unifying framework, show its properties, and describe the way it covers the two types of concept-forming operators. In fact, we show that the unifying framework covers a wider class of possible generalizations of TCA than just the two cases mentioned above. We also show that an analogy of the basic theorem of (dyadic) concept lattices holds true.

In the recent work on relational factor analysis [4,1,10] the (dyadic) concept lattices, both the antitone and the isotone, are of crucial importance because they provide optimal factors for relational matrix decompositions. In particular, triadic concept lattices were shown to be the space of optimal factors for factor analysis of three-way binary data in [12]. However, different types of a relational matrix decomposition (e.g. sup-min, min-t-norm decompositions) and the corresponding types of concept forming operators (e.g. isotone, antitone operators) are treated separately in the literature, although the results are very similar to each other. It is therefore desirable to find their suitable generalization. To develop a general mathematical framework for TCA in which all types of matrix decompositions are a specific instance of a more general decomposition is the main goal of this paper.

The paper is organized as follows: Section 2 recalls basic notions from (ordinary) triadic concept analysis, residuated lattices and fuzzy sets. Section 3 describes the unifying framework and its basic properties. In Section 4 we turn our attention to triadic concept-forming operators, triadic concepts, and concept trilattices. Section 5 considers reduction of the size of concept trilattices and shows that it is naturally included in the framework. Our conclusions and future research ideas are summarized in Section 6.

2. Preliminaries

2.1. Triadic concept analysis

This section introduces the basics of triadic concept analysis in ordinary case. For further information we refer to [23,33].

Triadic concept analysis is a method of exploratory data analysis that deals with ternary tabular data. Its input consists of a table that describes objects by binary attributes they have under various conditions, thus the input can be conceived as a ternary relation between a set of objects, a set of attributes, and a set of conditions.

Definition 1. A *triadic context* is a quadruple $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ where X_1, X_2, and X_3 are non-empty sets, and I is a ternary relation between them, i.e. $I \subseteq X_1 \times X_2 \times X_3$.

While the sets X_1, X_2, and X_3 are interpreted as the sets of objects, attributes, and conditions, respectively, I is interpreted as the incidence relation ("to have-under" relation). That is, $\langle x_1, x_2, x_3\rangle \in I$ is interpreted as: "object x_1 has attribute x_2 under condition x_3."

Let $\{i,j,k\} = \{1,2,3\}$ (i.e. $i,j,k \in \{1,2,3\}$ s.t. $i \neq j \neq k$). A triadic context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces three *concept-forming operators.* A concept-forming operator is a mapping assigning to pair of sets $C_i \subseteq X_i$ and $C_k \subseteq X_k$ the set $C_j \subseteq X_j$ given by

$$C_j = C_i^{(i,j,C_k)} = C_k^{(k,j,C_i)} = \{x_j \mid \langle x_i, x_j, x_k\rangle \in I \text{ for all } x_i \in C_i, x_k \in C_k\}.$$

Remark 1. We use the somewhat peculiar notation $C_i^{(i,j,C_k)}$ because of historical reasons. In [23,33] the concept-forming operators are induced in two steps. First, given a triadic context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ and a set $C_k \subseteq X_k$, one obtains a dyadic context $\langle X_i, X_j, I^{ij}_{C_k}\rangle$ given by

$$I^{ij}_{C_k} = \{\langle x_i, x_j\rangle \mid \langle x_i, x_j, x_k\rangle \in I \text{ for all } x_k \in C_k\}.$$

Then $C_i^{(i,j,C_k)}$ are just the dyadic concept-forming operators induced by $I^{ij}_{C_k}$. After a short inspection, we can see that for $C_i \subseteq X_i$ and $C_k \subseteq X_k$ we have $C_i^{(i,j,C_k)} = C_k^{(k,j,C_i)}$ (see also [7], Lemma 3.1).

The aim of TCA is to extract a hierarchically ordered set of clusters, called triadic concepts, from the triadic context. Triadic concepts are particular triplets $\langle A_1, A_2, A_3\rangle$ consisting of a subset of objects, a subset of attributes, and a subset of conditions. The sets A_1, A_2 and A_3 are maximal in the sense that A_1 is maximal set of objects having all attributes in A_2 under all conditions in A_3, and vice versa.

Definition 2. A *triadic concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle A_1, A_2, A_3\rangle$ of $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, and $A_3 \subseteq X_3$, such that for every $\{i,j,k\} = \{1,2,3\}$ we have $A_i = A_j^{(j,i,A_k)}$. The sets A_1, A_2, and A_3 are called the *extent*, *intent*, and *modus* of $\langle A_1, A_2, A_3\rangle$, respectively. The set of all triadic concepts of $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ (or just $\mathcal{T}(\mathbf{K})$ for short) and is called a *concept trilattice* of $\langle X_1, X_2, X_3, I\rangle$.

Let $\mathfrak{b}^w_{ik} : 2^{X_i} \times 2^{X_k} \to \mathcal{T}(\mathbf{K})$ be a map

$$\mathfrak{b}^w_{ik}(C_i, C_k) = \langle A_1, A_2, A_3\rangle, \tag{1}$$

which assigns to a pair of sets $C_i \subseteq X_i$ and $C_k \subseteq X_k$ a triple $\langle A_1, A_2, A_3\rangle$ such that $A_j = C_i^{(i,j,C_k)}$, $A_i = A_j^{(j,i,C_k)}$, and $A_k = A_i^{(i,k,A_j)}$. Then $\langle A_1, A_2, A_3\rangle$ is a triadic concept. That is, starting with a pair of sets C_i, C_k, one can obtain a triadic concept using concept-forming operators three times.

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