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A note on mean absolute deviation

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ABSTRACT

We introduce the idea of Mean Absolute Deviation (MAD). We provide a simple method for calculating the MAD that circumvents the use of the absolute value. We develop this both for the case of continuous and finite valued random variables. We introduce the idea of sub-mean and show how the MAD is related to the difference between the mean and sub-mean. We look at the MAD for random variables with symmetric probability density functions. We consider problem of determining the best estimate of a random variable under various measures of error.

1. Introduction

The variance and the Mean Absolute Deviation (MAD) provide measures of expected deviation of a random variable from its mean value [3,15,16]. The debate as to which is better has a long history going back to Edington's preference for the MAD in his pioneering work on astronomy [4] and Fisher's counter arguments [6]. Gorard [7] provides an elucidating discussion of this debate as well as the advantages of each. While the variance and its square root, the standard deviation, are much more widely used in applications than the mean absolute deviation, the MAD has been applied in financial applications [10–13] and other applications [1,5,8,9,17]. As discussed in [7] one reason for the less use of MAD is the unwieldiness of having to perform calculations involving the absolute value operations as required in the case of using mean absolute deviation. Here we suggest an expression for the mean absolute deviation that circumvents the use of the absolute value operation in calculating the mean absolute deviation. We feel that this simplified formulation for calculating the MAD will make it easier to use in applications.

2. Mean absolute deviation

Assume Y is a continuous random variable whose range is the interval $R = [a, b]$. We can associate with Y a non-negative function f called the probability density function such that $\text{Prob}(Y \in B) = \int_B f(y)dy$. We note that $\text{Prob}(Y \in R] = \int_a^b f(y)dy = 1$.

For any $r \in R$ the function

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$$F(r) = \text{Prob}(Y \leqslant r) = \int_a^r f(y)dy$$

is called its cumulate distribution function, CDF. We can associate with Y an expected value or mean defined as $E(Y) = \mu = \int_a^b yf(y)dy$. It is known that $\mu \in [a,b]$. The variance is a measure of deviation from the mean defined as $\text{Var}(Y) = \sigma^2 = E((Y-\mu)^2) = \int_a^b (y-\mu)^2 f(y)dy$. It is well known that Var $(Y) = E(Y)^2 - \mu^2$ closely related to the variance is the standard deviation, σ, equal to the square root of the variance.

Another measure of deviation is the mean absolute direction, MAD, it is defined

$$\text{MAD}(Y) = E(|y-\mu|) = \int_a^b |y-\mu|\, f(y)\, dy$$

One encumbrance that arises when using the MAD is that we must deal with the absolute value. In the following we shall provide an expression for the MAD that avoids the use of the absolute value and provides a relatively simple expression for calculating the MAD.

Since $|y-\mu| = \mu - y$ for $y \leqslant \mu$ and $|y-\mu| = y - \mu$ for $y \geqslant \mu$ we can express

$$\text{MAD}(Y) = \int_a^\mu (\mu - y)\, f(y)dy + \int_\mu^b (y-\mu)\, f(y)dy.$$

We observe that

$$\text{MAD}(Y) = \int_a^\mu \mu f(y)dy - \int_\mu^b \mu f(y)dy - \int_a^\mu yf(y)dy + \int_\mu^b yf(y)dy.$$

We further observe that

$$\int_\mu^b \mu f(y)dy = \int_a^b \mu f(y)dy - \int_a^\mu \mu f(y)dy = \mu - \int_a^\mu \mu f(y)dy$$

and that

$$\int_\mu^b yf(y)dy = \int_a^b yf(y)dy - \int_a^\mu yf(y)dy = \mu - \int_a^\mu yf(y)dy$$

Placing these equivalences into our definition for MAD(Y) we get

$$\text{MAD}(Y) = \int_a^\mu \mu f(y)dy - \mu + \int_a^\mu \mu\, f(y)dy - \int_a^\mu y\, f(y)dy + \mu - \int_a^\mu y\, f(y)dy$$

$$\text{MAD}(Y) = 2\int_a^\mu \mu\, f(y)dy - 2\int_a^\mu y\, f(y)dy = 2\int_a^\mu (\mu - y)f(y)dy$$

The above provides a very simple form for calculating the MAD. At times we shall refer to this expression of mean absolute direction as EMAD for easily MAD.

Let us look at some classic probability distributions. Consider a uniform distribution

$$f(x) = \tfrac{1}{b-a} \quad \text{for } x \in [a,b]$$
$$f(x) = 0 \quad \text{elsewhere}$$

For this the mean $\mu = \frac{a+b}{2}$ and hence

$$\text{MAD}(Y) = 2\int_a^{(a+b)/2} \left(\frac{a+b}{2} - y\right)\frac{1}{b-a}dy = \frac{1}{b-a}\int_a^{(a+b)/2} (a+b-2y)dy$$

$$\text{MAD}(Y) = \frac{1}{b-a}(ay+by-y^2) \quad |a^{(a+b)/2} = b-a\left[\frac{(a+b)}{2}a + b\frac{(a+b)}{2} - \frac{(a+b)^2}{2} - (a^2+ab-a^2)\right.$$

$$\text{MAD}(Y) = \frac{1}{4}(b-a)$$

We note that the variance of the uniform distribution is $\frac{(b-a)^2}{12}$ and its standard deviation $\sigma = \frac{b-a}{\sqrt{12}} > \frac{b-a}{4}$.

Consider now the exponential type distribution where

$$f(y) = \lambda e^{-\lambda y} \quad \text{for } y \geqslant 0$$
$$f(y) = 0 \quad \text{for } y < 0$$

For this distribution it is well known [16] that $\mu = 1/\lambda$. In this case

$$\text{MAD}(Y) = 2\int_0^{1/\lambda} \left(\frac{1}{\lambda} - y\right)\lambda e^{-\lambda y} = 2\int_0^{1/\lambda} (e^{-\lambda y} - \lambda y e^{-\lambda y})dy$$

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