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Decision-theoretic three-way approximations of fuzzy sets

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ABSTRACT

A three-way, three-valued, or three-region approximation of a fuzzy set is constructed from a pair of thresholds (α, β) on the fuzzy membership function. An element whose membership grade equals to or is greater than α is put into the positive region, an element whose membership grade equals to or is less than β is put into the negative region, and an element whose membership grade is between β and α is put into the boundary region. A fundamental issue is the determination and interpretation of the required pair of thresholds. In the framework of shadowed sets (i.e., an example of three-way approximations of fuzzy sets), Pedrycz provides an analytic solution to computing the thresholds by searching for a balance of uncertainty introduced by the three regions. To gain further insights into three-way approximations of fuzzy sets, we introduce an alternative decision-theoretic formulation in which the required thresholds are computed by minimizing decision cost.

1. Introduction

Fuzzy sets extend the classical sets by allowing graded membership values so that one can describe a concept that has an unsharp, gradually changing boundary [41]. The use of the unit interval $[0, 1]$ as the set of membership grades has both advantages and shortcomings. On the one hand, an infinite set of values provides a high degree of flexibility and a great expressive power mathematically. One can distinguish objects at minute details by using many levels of membership grade. On the other hand, the infinity number of values leads to a difficulty in interpreting and understanding a fuzzy membership function in practice. In many situations, our perception of fuzziness is of a qualitative nature. We may be only able to distinguish objects by using a few levels of fuzziness. A classical study of Miller [19] shows that human can only process about seven plus or minus two units of information. More recent studies suggest that the actual number is smaller and is around four [8]. The use of a few grade levels has cognitive advantages. In practical applications, it may also happen that we do not need to distinguish objects that are very similar to each other. An approximation may be sufficient, as a very precise membership value may not offer much more significant information. In addition, estimating and representing uncertainty or vagueness by a numeric value usually associate with a cost of observation or an error of estimation. Obtaining a more precise numeric value usually leads to a higher cost although a lower error. There is a trade-off between accuracy and cost. Accordingly, approximating a fuzzy set by using several levels of grade is of practical importance. An approximation of a fuzzy set by using only a few grade values may simplify a problem and, more often than not, is practically sufficient.

The concept of shadowed sets, proposed by Pedrycz [23,26–28], is an example of three-way, three-valued, or three-region approximations of a fuzzy set. It has received much attention in recent years in theory and applications

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[2,4,7,12–15,18,20,22–24,30–32,40,43,44]. Intuitively, the construction of a shadowed set is based on the following principles. If the membership grade of an element is close to 1, it would be considered to be the same as 1 and is elevated to 1; if the membership grade is close to 0, it would be considered to be the same as 0 and is reduced to 0; if the membership grade is neither close to 1 nor close to 0, it would be put into a shadowed region. The elevation and reduction operations use thresholds that provide semantically meaningful and acceptable levels of degree of closeness of membership value to 1 and 0, respectively.

A fundamental issue of three-way approximations of fuzzy sets is the interpretation and determination of thresholds. In searching of the optimal thresholds, Pedrycz [23] suggests a method by minimizing an objective function that provides a balance of uncertainty characterized by a fuzzy set. More specifically, the objective function is expressed as the difference between the shadowed area and the sum of the increased and decreased areas by the elevation and reduction operations. The approach works well computationally. On the other hand, we can improve the method from several aspects. The minimization procedure provides an analytic solution when the thresholds are related, i.e., they add up to 1. For a pair of unrelated thresholds, one may not have an analytic solution. Pedrycz's objective function is one of many possible ways to interpret and compute thresholds in terms of uncertainty. It is more useful to investigate the physical meaning of various objective functions based on more operational notions such as error rate and cost. By applying results from decision-theoretic rough sets (DTRS) [39] and three-way decisions [35–37], this paper solves these difficulties by introducing a model of decision-theoretic three-way approximations of fuzzy sets.

A basic idea of three-way decisions is to classify a set of objects into three regions, called the positive, negative and boundary regions, by using an evaluation function and a pair of thresholds [37]. If we interpret the membership function of a fuzzy set as an evaluation, we can immediately obtain a new interpretation of a three-way approximation of a fuzzy set. For each object, one can make one of three decisions: elevate the membership grade to 1, reduce the membership grade to 0, or change the membership grade to a third intermediate value. The elevation of a membership grade to 1 means that one accepts an object to be an instance of the concept represented by a fuzzy set, as its membership grade is close to 1. The reduction of a membership grade to 0 means that one rejects the object to be an instance of the concept, as its membership grade is close to 0. For an object whose membership grade is neither close to 1 nor close to 0, one would like to make a noncommitment decision by using a third value. Each decision is associated with some errors and costs. Such a three-way decision interpretation offers naturally a model for constructing three-way approximations by using the notions of error and cost. One can obtain an optimal pair of thresholds by minimizing overall error or cost of three-way approximations.

The rest of this paper is organized as follows. Section 2 reviews basic concepts of fuzzy sets, three-way approximations of fuzzy sets and shadowed sets from the viewpoint of three-way decisions. Section 3 proposes an error-based interpretation of shadowed sets. It shows the limitations of the existing formulation of shadowed sets and, therefore, motivates the introduction of a decision-theoretic model. Section 4 is a detailed derivation of the decision-theoretic model of three-way approximations of fuzzy sets. Section 5 discusses two special models, namely, the error-based $(0.75, 0.25)$-model and a symmetric $(\alpha, 1-\alpha)$-model.

2. Approximations of fuzzy sets with three-way decisions

This section reviews basic concepts of fuzzy sets, three-way approximations of fuzzy sets and shadowed sets within the framework of three-way decisions.

2.1. Three-way decisions

By observing a common practice of decision making across many disciplines, Yao [37] proposes a general framework of three-way decisions. Suppose U is a universal set of objects and C is a set of conditions called criteria. An object in U satisfies, does not satisfy the criteria or only satisfies the criteria to a certain degree. Three-way decisions deal with the classification of objects based on the set of criteria. According to an evaluation of the satisfiability of objects, one can make three-way decisions as follows:

1. Accept an object as satisfying the set of criteria if its degree of satisfiability is at or above a certain level.
2. Reject the object by treating it as not satisfying the criteria if its degree of satisfiability is at or below another level.
3. Neither accept nor reject the object but opt for a noncommitment or deferment decision that needs further investigation.

With three-way decisions, objects are classified into three regions, called the positive, negative and boundary regions, respectively. Three-way decisions may be considered as a generalization of two-way decisions. The introduction of a third option of noncommitment leads to more flexibility. Intuitively, the set of criteria and the notion of satisfiability may be interpreted by using more practical and operational concepts of, for example, costs, risks, profits, rates of error and so on. The values of the evaluation are the degrees of satisfiability or desirability of objects with respect to the set of conditions. For this purpose, the set of values must be equipped with an order relation. In this paper, we adopt a special model of three-way decisions that uses a totally ordered set in which any two elements are comparable [37].

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