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On equivalence of conceptual scaling and generalized one-sided concept lattices

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A B S T R A C T

The methods of conceptual scaling and generalized one-sided concept lattices represent different possibilities on how to deal with many-valued contexts. We briefly describe these methods and prove that they are equivalent. In particular, we show that the application of these two approaches to a given many-valued context yields the same closure system on the set of all objects. Based on this equivalence, we propose a possible attribute reduction of one-sided formal contexts.

1. Introduction

Formal Concept Analysis (FCA) is a theory of data analysis for identification of conceptual structures among data sets. Mathematical theory of FCA is based on the notion of concept lattices and is well developed in the monograph of Ganter and Wille [11]. In practice, there are natural examples of object-attribute models where relationships between objects and attributes are represented by many-valued relations. Therefore several attempts to modify FCA for these situations have been proposed. One of the ways how to handle with many-valued contexts is known as a process of conceptual scaling described in [11]. In conceptual scaling, a given many-valued context is transformed into a classical one by introducing new binary attributes and consequently the concept lattice is defined within this framework.

Another way to deal with non-crisp data in object-attribute models is an attempt to fuzzify the FCA method. We mention an approach of Bělohlávek [2–5] based on the logical framework of complete residuated lattices, work of Georgescu and Popescu to extend this framework to non-commutative logic [12–14], an approach of Krajči [18], Popescu [30], work on multi-adjoint concept lattices [21–25] and other approaches [1,8,27–29,31]. A nice survey and comparison of some existing approaches to fuzzy concept lattices is presented in [7,6].

A special case of fuzzy FCA is one-sided concept lattice, where usually objects are considered as crisp subsets and attributes obtain fuzzy values. From existing one-sided approaches we mention papers of Krajči [17], Ben Yahia and Jaoua [9], work of Jaoua and Elloumi on Galois lattices of real relations [16].

Recently there was a generalization of all these approaches, so called generalized one-sided concept lattices, cf. [10,27] for more details. This type of one-sided concept lattices allows to consider different types of structure for truth degrees

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(represented by complete lattices). The main goal of this paper is to prove that the methods of conceptual scaling and generalized one-sided concept lattices are equivalent. We show that the application of these two methods to a given many-valued context yields the same closure system on the set of all objects.

Since the theory of concept lattices is closely related to Galois connections and closure systems, in the section of Preliminaries we give a brief overview of these notions. We also describe the method of conceptual scaling, we illustrate this method on a simple example and finally we give basic definitions and results concerning generalized one-sided concept lattices. In Section 3 we describe the transformation of many-valued context to the generalized one-sided context. Next we prove our main result, i.e., the equivalence of these two approaches. In the final section we describe a possible application of the proved result. In particular, the problem of the reduction of generalized one-sided formal context is dealt with. We describe a general method on how to reduce attributes and their truth value structures. Finally we prove the isomorphism theorem, which shows that this reduction yields isomorphic concept lattices.

2. Preliminaries

In this section we give a basic overview of the algebraic notions needed for our purposes. In the sequel we assume that the reader is familiar with the basic notions of FCA and lattice theory. We will use the standard terminology and the notation as in [11,15], respectively.

Definition 2.1. Let $(P, \leqslant)$ and $(Q, \leqslant)$ be ordered sets and let

$$\varphi: P \to Q \quad \text{and} \quad \psi: Q \to P$$

be two mappings between these ordered sets. Such a pair (φ, ψ) of mappings is called a *Galois connection* between the ordered sets if:

(a) $p_1 \leqslant p_2$ implies $\varphi(p_1) \geqslant \varphi(p_2)$,
(b) $q_1 \leqslant q_2$ implies $\psi(q_1) \geqslant \psi(q_2)$,
(c) $p \leqslant \psi(\varphi(p))$ and $q \leqslant \varphi(\psi(q))$.
These two mappings are also called *dually adjoint* to each other. We note that

$$\varphi = \varphi \circ \psi \circ \varphi \quad \text{and} \quad \psi = \psi \circ \varphi \circ \psi$$

and that the conditions (a)–(c) are equivalent to the following one:
(d) $p \leqslant \psi(q)$ if and only if $\varphi(p) \geqslant q$.

Throughout this paper we will denote the class of all partially ordered sets by the symbol Pos and the class of all complete lattices will be denoted by CL.

Galois connections between complete lattices are closely related to the notions of closure operator and closure system. Let L be a complete lattice. By a *closure operator* in L we understand a mapping c: $L \to L$ satisfying:

(a) $x \leqslant c(x)$ for all $x \in L$,
(b) $c(x_1) \leqslant c(x_2)$ for $x_1 \leqslant x_2$,
(c) $c(c(x)) = c(x)$ for all $x \in L$ (i.e., c is idempotent).

A subset X of the complete lattice L is called *closure system* in L if X is closed under arbitrary meets. We note that this condition guarantees that $(X, \leqslant)$ is a complete lattice, in which the infima are the same as in L, but the suprema in X may not coincide with those from L. For a closure operator c in L, the set $\mathsf{FP}(c)$ of all fixed points of c (i.e., $\mathsf{FP}(c) = \{x \in L: c(x) = x\}$) is a closure system in L. Conversely, for closure system X in L, mapping C_X: $L \to L$ defined by $\mathsf{C}_X(x) = \bigwedge\{u \in X: x \leqslant u\}$ is a closure operator in L. Moreover these correspondences are inverse of each other, i.e., $\mathsf{FP}(\mathsf{C}_X) = X$ for each closure system X in L and $\mathsf{C}_{\mathsf{FP}(c)} = c$ for each closure operator c in L.

The dual notion is *interior system*. A subset Y of a complete lattice L is called an interior system in L if Y is closed under arbitrary joins. This condition also guarantees that $(Y, \leqslant)$ is complete lattice, in which suprema are the same as in L, but infima in Y may not coincide with those from L.

There is a well known relationship between the closure operators induced by the Galois connections [26]. Two ordered sets P, Q are called *isomorphic (dually isomorphic)*, if there is an order preserving (order reversing) bijective mapping f: $P \to Q$ such that f^{-1} is also order preserving (order reversing). Let $L, M \in \mathsf{CL}$ and (φ, ψ) be Galois connection between L and M. Then mapping $\varphi \circ \psi$: $L \to L$ is closure operator in L, similarly, $\psi \circ \varphi$: $M \to M$ is closure operator in M. Moreover the corresponding closure systems are dually isomorphic.

Conversely, suppose that X_1 and X_2 are closure systems in L, M respectively, and f: $X_1 \to X_2$ is a dual isomorphism between complete lattices $(X_1, \leqslant)$ and $(X_2, \leqslant)$. Then a pair $(c_{X_1} \circ f, c_{X_2} \circ f^{-1})$, where c_{X_1}, c_{X_2} are closure operators corresponding to X_1 and to X_2, forms a Galois connection between L and M. Hence, any Galois connection between complete lattices induces dually isomorphic closure systems on these lattices and any two dually isomorphic closure systems define a Galois connection.