International Journal of Approximate Reasoning 101 (2018) 255–275

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Dicovering approximation spaces and definability ☆

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ARTICLE INFO

Article history:
Received 26 January 2018
Received in revised form 29 May 2018
Accepted 20 July 2018
Available online 1 August 2018

Keywords:
Approximation space
Covering
Definable set
Dicovering
Rough set
Texture

ABSTRACT

In this work, we define a category **Cap** of covering approximation spaces whose morphisms are functions satisfying a refinement property. We give the relations among **Cap**, and the category **Top** of topological spaces and continuous functions, and the category **Rere** of reflexive approximation spaces and the relation preserving functions. Further, we discuss the textural versions **diCap**, **dfDitop** and **diRere** of these categories. Then we study the definability in **Cap** with respect to five covering-based approximation operators. In particular, it is observed that via the morphisms of **Cap**, we may get more information about the subsets of the universe.

1. Introduction

Textural arguments on rough sets provide remarkable observations for some fundamental concepts of mathematics. For instance, the concepts of relation and function can be stated in terms of definability in rough set theory. A relation is a function if and only if every singleton subset is definable with respect to successor neighborhood operator. Further, a function r has an inverse if and only if every singleton subset is definable with respect to predecessor neighborhood operator [14,17]. By a textural argument, it may be also observed that the lower and upper approximations of Yao can be stated by point-free formulations [10]. Recall that an approximation space in the sense of Pawlak is a pair (U, r) where U is the universe and r is an equivalence relation on U [27]. In a more general setting, an approximation space can be defined using a cover of the universe. A pair $(U, \mathcal{C})$ is called *a covering approximation space* if $\mathcal{C}$ is a cover of the universe U [4]. Pomykala, Yao and, Zhu and Wang studied the covering approximation spaces taking the upper approximation of Zakowski and its dual as a lower approximation operator [28,34,41]. An extensive study on covering approximation spaces can be found in the recent paper of Yao and Yao in [36]. Zhang and Luo proved that the first five covering-based approximations can be stated using approximation operators with respect successor neighborhoods of reflexive relations [38]. This may provide more information for a subset of the universe using a comparison between the definabilities of subsets with respect to approximation operators and neighborhoods.

A texturing $\mathcal{U}$ is a family of subsets of the given universe U subjected to certain conditions and the pair $(U, \mathcal{U})$ is called *a texture space* or in brief *a texture* [5]. Essentially, the basic motivation of textures is to provide a point-set based setting for fuzzy sets [6]. However, textures can be also adapted to obtain a general approach to rough set systems considering the

☆ This work has been supported by Hacettepe University Scientific Research Projects Coordination Unit (FBB-2017-16209).
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https://doi.org/10.1016/j.ijar.2018.07.009
0888-613X/

granular operator spaces given by Mani in [24]. For the discussions on rough sets in the framework of textures, we refer to [10–18,31,32].

A dicovering $\mathcal{H}$ of a texture $(U, \mathcal{U})$ is a natural counterpart of a cover in textures. Then the triple $(U, \mathcal{U}, \mathcal{H})$ is called *a dicovering approximation space*. A discussion on the connections between covering and dicovering approximation spaces are extensively given in [16]. In this work, we continue study of the theory of covering approximation spaces using textures. Section 2 is devoted to the basic concepts and results on textures. The first argument on rough sets in the framework of category theory was given by Banerjee and Chakraborty in [2]. Further studies on rough sets with respect to category theory can be found in [3,13,14,19,22,23,25]. In Section 3, we present a categorical discussion with respect to granular computing considering dicoverings in textures. We show that dicovering approximation spaces and difunctions $(f, F): (U, \mathcal{U}, \mathcal{H}) \to (V, \mathcal{V}, \mathcal{D})$ satisfying the condition $(f, F)^{\to}\mathcal{H} \prec \mathcal{D}$ form a category denoted by **diCap**. Recall that reflexive approximation spaces and relation preserving functions also form a category denoted by **Rere** [1]. A textural version of **Rere** is the category **diRere** of reflexive textural approximation spaces and direlation preserving difunctions [14]. Here, we construct two functors $\mathfrak{D}$: **diCap** → **diRere** and $\mathfrak{M}$: **diRere** → **dfDitop** where **dfDitop** is the textural version of the category **Top** of topological spaces and continuous functions [9]. In Section 4, we define the category **Cap** of covering approximation spaces whose morphisms are functions $f : (U, \mathcal{C}) \to (V, \mathcal{A})$ satisfying the property $f(\mathcal{C}) \prec \mathcal{A}$. We give the connections among the categories **Top**, **Rere**, **Cap** and the textural versions **dfDitop**, **diRere** and **diCap**, respectively. In the last section, we investigate the definability with respect to the first five covering-based approximation operators in the category **Cap**.

2. Preliminaries

For the motivation, and the concepts and results on textures which are not explained in this paper, we refer to [5,6,8]. For the basic categorical results and terminology, we refer to [1].

Basic concepts

Let U be a set. Then $\mathcal{U} \subseteq \mathcal{P}(U)$ is called a *texturing* of U, and $(U, \mathcal{U})$ is called a *texture space*, or simply a *texture*, if

(i) $(\mathcal{U}, \subseteq)$ is a complete lattice containing U and $\emptyset$, which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions,
(ii) $\mathcal{U}$ is completely distributive.
(iii) $\mathcal{U}$ separates the points of U. That is, given $u_1 \neq u_2$ in U there exists $A \in \mathcal{U}$ such that $u_1 \in A$, $u_2 \notin A$, or $u_2 \in A$, $u_1 \notin A$.

Textures need not be closed under ordinary set complementation. *A complementation* on a texture $(U, \mathcal{U})$, is a mapping $c_U : \mathcal{U} \to \mathcal{U}$ satisfying the conditions

$$\forall A \in \mathcal{U}, c_U(c_U(A)) = A, \text{ and}$$
$$\forall A, B \in \mathcal{U}, A \subseteq B \Longrightarrow c_U(B) \subseteq c_U(A).$$

Then the triple $(U, \mathcal{U}, c_U)$ is called *a complemented texture space*. For $u \in U$, the p-sets and q-sets are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \text{ and}$$
$$Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}.$$

The following are canonical examples for textures:

Discrete texture

For the universe U, the family $\mathcal{P}(U) = \{A \mid A \subseteq U\}$ is a texturing on U. The pair $(U, \mathcal{P}(U))$ is called *a discrete texture*. For $u \in U$, we clearly have $P_u = \{u\}$ and $Q_u = U \setminus \{u\}$ and the mapping $c_U : \mathcal{P}(U) \to \mathcal{P}(U)$ is the ordinary complementation on $(U, \mathcal{P}(U))$ defined by $c_U(A) = U \setminus A$ for all $A \in \mathcal{P}(U)$.

Fuzzy texture

The family $\mathcal{M} = \{(0, r] \mid r \in [0, 1]\}$ is a complemented texture on $M = (0, 1]$ which is called *the fuzzy texture*. Here, we have $P_r = Q_r = (0, r]$ for all $r \in (0, 1]$. The complementation $c_M : \mathcal{M} \to \mathcal{M}$ is defined by $\forall r \in (0, 1]$, $c_M(0, r] = (0, 1 - r]$.

Products of textures

The product of a family of textures and its basic properties can be found extensively in [6,9]. In fact, the morphisms (direlations) between any two textures $(U_1, \mathcal{U}_1)$ and $(U_2, \mathcal{U}_2)$ are the pairs where the compounds are the elements of the product of a discrete texture $(U_1, \mathcal{P}(U_1))$ and a texture $(U_2, \mathcal{U}_2)$. Therefore, for the sake of simplicity, we consider here the product of two textures. Now let us take the family $\mathcal{A} = \{A \times U_2 \mid A \in \mathcal{U}_1\} \bigcup \{U_1 \times B \mid B \in \mathcal{U}_2\}$ and define

$$\mathcal{B} = \{\bigcup_{j \in J} E_j \mid \{E_j\}_{j \in J} \subseteq \mathcal{A} \text{ and } J \text{ is an index set}\}.$$