JID:IJA AID:8227 /FLA [m3G; v1.240; Prn:2/07/2018; 11:25] P.1 (1-20)

International Journal of Approximate Reasoning ••• (••••) •••–•••

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Linear criterion for testing the extremity of an exact game based on its finest min-representation ☆

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ARTICLE INFO

Article history:
Received 2 November 2017
Received in revised form 19 June 2018
Accepted 26 June 2018
Available online xxxx

Keywords:
Extreme exact game
Coherent lower probability
Core
Supermodular game
Finest min-representation
Oxytrophic game

ABSTRACT

A game-theoretical concept of an exact (cooperative) game corresponds to the notion of a discrete coherent lower probability, used in the context of imprecise probabilities. The collection of (suitably standardized) exact games forms a pointed polyhedral cone and the paper is devoted to the recognition of extreme rays of that cone, whose generators are called *extreme exact games*. We give a necessary and sufficient condition for an exact game to be extreme. Our criterion leads to solving a simple linear equation system determined by a certain min-representation of the game. It has been implemented on a computer and a web-based platform for testing the extremity of an exact game is available, which works with a modest number of variables.

The paper also deals with different *min-representations* of a fixed exact game μ, which can be compared with the help of the concept of a *tightness structure* (of a min-representation) introduced in the paper. The collection of tightness structures (of min-representations of μ) is shown to be a finite lattice with respect to a refinement relation. We give a method to obtain a min-representation with the finest tightness structure, which construction comes from the coarsest standard min-representation of μ given by the (complete) list of vertices of the core (polytope) of μ. The newly introduced criterion for exact extremity is based on the finest tightness structure.

1. Motivation and overview of former results

The notion of a *coherent lower probability* and that of an induced *credal set* (of discrete probability distributions) are traditional topics of interest in the theory of imprecise probabilities. These notions correspond to game-theoretical concepts of an *exact game* and its *core* (polytope), widely used in the context of cooperative coalition games. The analogy is even broader: a lower probability avoiding sure loss corresponds to a weaker concept of a balanced game while a *2-monotone lower probability* (= capacity) corresponds to a stronger concept of a *supermodular game*, also named a convex game in game-theoretic community.

The discrete case is considered here: the sample space (= frame of discernment) for distributions is a fixed finite set N having at least two elements. The elements of N correspond to players in the context of cooperative game theory and to random variables in yet another context of probabilistic conditional independence structures. The collection of coherent lower probabilities on N, where $n = |N|$, is a polytope in a 2^n-dimensional real vector space, while the set of non-negative

☆ This paper is part of the Virtual special issue on Tenth International Symposium on Imprecise Probability: Theories and Applications (ISIPTA '17), Edited by Alessandro Antonucci, Giorgio Corani, Inés Couso and Sébastien Destercke.

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https://doi.org/10.1016/j.ijar.2018.06.007
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Please cite this article in press as: M. Studený, V. Kratochvíl, Linear criterion for testing the extremity of an exact game based on its finest min-representation, Int. J. Approx. Reason. (2018), https://doi.org/10.1016/j.ijar.2018.06.007

exact games is a pointed polyhedral cone whose extreme rays are generated just by extreme points of that polytope. In fact, the set of coherent lower probabilities can be viewed as the intersection of the cone of non-negative exact games μ with a normalizing hyperplane specified by $\mu(N) = 1$.

This paper offers a method to recognize whether a ray is extreme in the cone of exact games, which implicitly gives a method to recognize extreme coherent lower probabilities. From the geometric point of view, the problem of recognition of extreme rays is interesting in itself. Thus, it is natural that it has been treated earlier, both in the context of game theory and in the context of imprecise probabilities.

1.1. Extremity criteria in game theory

Some effort to develop criteria to recognize the extremity of an exact game was exerted earlier by Rosenmüller [16, §4 of chapter 5] in his book on game theory. He offered one necessary and one sufficient condition for the extremity based on a *min-representation* of the exact game; however, these conditions have a limited scope of use because they are applicable only in quite special situations. In this paper we follow the idea of a min-representation and propose a more general criterion whose input is the list of vertices of the *core*, which serves as a *standard min-representation* of any exact game. Our condition is necessary and sufficient for the extremity of an exact game. Specifically, our extremity criterion can use any of the so-called *finest min-representations* of an exact game; such a min-representation can easily be obtained from the standard one. Note that, for certain exact games, the standard min-representation is the only (finest) min-representation. These games were named *oxytrophic* in [16].

1.2. Extremity in context of imprecise probabilities

Analogous problems were studied in connection with imprecise probabilities. Questions raised by Maass [11] motivated Quaeghebeur and de Cooman [14] to become interested in *extreme (coherent) lower probabilities* and to compute these in the case of small $n = |N|$. Antonucci and Cuzzolin [1] considered an enlarging transformation of a credal set with a finite number of extreme points, when the respective (coherent) lower probability is computed and then a larger credal set is induced by the lower probability. Their second step, namely representing a coherent lower probability by the vertices of the induced credal set, corresponds to our standard min-representation of an exact game.

The theme of characterizing the extreme lower probabilities from [14] motivated an even more general problem of characterizing the *extreme lower previsions* discussed by De Bock and de Cooman [2]; they related extreme lower previsions to indecomposable compact convex sets in a finite-dimensional space.

It is always useful to be aware of the correspondence between concepts from different areas. For instance, Wallner [21] confirmed a conjecture raised by Weichselberger that the credal set induced by a (coherent) lower probability has at most $n!$ vertices. Nonetheless, the same result was achieved already by Derks and Kuipers [6] in the context of cooperative game theory. They also made an interesting observation that whenever a core of an exact game has $n!$ vertices then it has the maximal number of $2^n - 2$ facets and gave an example of a game in the relative interior of the exact cone whose respective core does not have the maximal number of $n!$ vertices.

Note in this context that the polytopes which are cores of exact games are even more general than the so-called generalized permutohedra introduced in [13], which are also known to have at most $n!$ vertices; see [19, Remark 12].

1.3. The case of the supermodular cone

The criterion we offer here is a modification of the criterion from [19], where a necessary and sufficient condition was provided for a supermodular game to be extreme in the cone of (suitably standardized) supermodular games. That result was motivated by the research on conditional independence structures [17], in which context extreme supermodular games encode submaximal structural conditional independence models. The supermodular criterion leads to solving a simple linear equation system determined by certain combinatorial structure (of the core), which concept was pinpointed earlier by Kuipers et al. [9]. An analogous combinatorial concept has also appeared in the context of imprecise probabilities: Bronevich and Rozenberg [3, Proposition 5] in their description of non-extreme 2-monotone lower probabilities use “the collection of maximal lattices on which the 2-monotone measure is additive”, which collection seems to coincide with the above-mentioned concept of *core structure* from [9,19]. More specifically, on basis of our consultation of [3], we think that such coincidence holds but a complete proof of this conjecture of ours would require some work because those two concepts are defined in different terms.

1.4. The case of the exact cone

The criterion for the extremity in the exact cone from the present paper can be viewed as a generalization of the former supermodular criterion [19]. One can assign an analogous linear equation system to any sensible min-representation of an exact game. The equation system is determined by the *tightness structure* of the min-representation, which is a combinatorial concept directly generalizing that of a core structure from [9,19]. Nevertheless, in the case of a general min-representation some modification is needed. More specifically, the equation system assigned to a general min-representation can have some non-zero dummy solutions, which never occur in case of the standard min-representation. Fortunately, the dimension of the