International Journal of Approximate Reasoning 101 (2018) 1–9

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

# Logical foundation of the quintuple implication inference methods ☆

Minxia Luo *, Kaiyan Zhou

*Department of Information and Computing Science, China Jiliang University, Hangzhou 310018, PR China*

ARTICLE INFO

*Article history:*
Received 5 December 2017
Received in revised form 2 June 2018
Accepted 7 June 2018
Available online 14 June 2018

*Keywords:*
Fuzzy reasoning
Monoidal t-norm based fuzzy logic
Quintuple implication inference methods

ABSTRACT

The quintuple implication inference methods for fuzzy reasoning characterizes the solution $B^*(A^*)$ for the fuzzy modus ponens (fuzzy modus tollens) as the formula that is best supported by $A \to B$, $A^* \to A$ and $A^*(A \to B, B \to B^*$ and $B^*)$. In this study, we provide a predicate formal representation of the solution for the quintuple implication inference methods based on the many-sorted first-order formal system Monoidal t-norm based logic MTL$\forall_{ms}$, including detailed logic proofs. We bring the quintuple implication inference methods within a logical framework and provide a sound logic foundation for the quintuple implication inference methods of fuzzy reasoning.



## 1. Introduction

Fuzzy reasoning includes many important inference methods for implementing fuzziness where the fundamental aim is to imitate the human inference mechanism. Fuzzy reasoning has achieved great success as the core of fuzzy control. In fuzzy reasoning, the most fundamental patterns are fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT), which can be expressed as follows (see [10,11]).

FMP$(A, B, A^*)$. For a given rule $A \to B$ and premise $A^*$, calculate the fuzzy consequent $B^*$.
FMT$(A, B, B^*)$. For a given rule $A \to B$ and premise $B^*$, calculate the fuzzy consequent $A^*$.

In order to solve these problems, the compositional rule of inference (CRI) was proposed by Zadeh [10], where the CRI solutions for FMP and FMT are as follows:

$$\text{CRI solution of FMP:}\ \ B^*(y) = \bigvee_{x \in X} (A^*(x) \wedge R(x, y))$$
$$\text{CRI solution of FMT:}\ \ A^*(x) = \bigvee_{y \in Y} (B^*(y) \wedge R(x, y)),$$

where $R(x, y) = (1 - A(x)) \vee (A(x) \wedge B(y))$.

The CRI method has some advantageous properties such as simple calculations and natural forms but as noted by Wang [8], the CRI method lacks a clear logical meaning and is not reductive. Thus, the full implication triple I (TI) method was

☆ This study was supported by the National Natural Science Foundation of China (Grant Nos. 61773019 and 11701540).
* Corresponding author at: School of Sciences, China Jiliang University, Hangzhou 310018, PR China.
*E-mail address:* minxialuo@163.com (M. Luo).

https://doi.org/10.1016/j.ijar.2018.06.001
0888-613X/

proposed by Wang [9], to bring fuzzy reasoning within the framework of semantic implication. The TI solutions for FMP and FMT are as follows:

$$\text{TI solution of FMP: } B^*(y) = \bigvee_{x\in X} (A^*(x) \otimes (A(x) \to B(y)))$$
$$\text{TI solution of FMT: } A^*(x) = \bigwedge_{y\in Y} ((A(x) \to B(y)) \to B^*(y))$$

However, in both the CRI and TI methods, the closeness between $A$ and $A^*$ (or $B$ and $B^*$) is not explicitly considered when calculating the consequence, which may make the computed solution incorrect or misleading in some cases. Therefore, the quintuple implication (QI) method for fuzzy reasoning was proposed by Zhou et al. [12] to characterize the solution $B^*(A^*)$ of FMP (FMT) as the formula that is best supported by $A \to B$, $A^* \to A$ and $A^*(A \to B, B \to B^*$ and $A$). Theorem 2.1 gives the form of the expression. As shown in the following example, the QI method for fuzzy reasoning is considered to be more reasonable in the fuzzy reasoning process and it accords better with human thinking.

For example [12], let the universe X=$\{1, 2, 3, 4, 5\}$. Suppose that small, medium, and large are three fuzzy sets on $X$, which are defined as follows:

$$[small] = \tfrac{1}{1} + \tfrac{0.3}{2} + \tfrac{0}{3} + \tfrac{0}{4} + \tfrac{0}{5},$$
$$[large] = \tfrac{0}{1} + \tfrac{0}{2} + \tfrac{0}{3} + \tfrac{0.3}{4} + \tfrac{1}{5},$$
$$[medium] = \tfrac{0}{1} + \tfrac{0.3}{2} + \tfrac{1}{3} + \tfrac{0.3}{4} + \tfrac{0}{5}.$$

For simplicity, we denote [small] $= [1, 0.3, 0, 0, 0]$, [large] $= [0, 0, 0, 0.3, 1]$, and [medium] $= [0, 0.3, 1, 0.3, 0]$. The problem for FMP is stated as follows. Let $A(x)$ denote "$x$ is small," $B(y)$ denotes "$y$ is large," and $A^*(x)$ denotes "$x$ is medium." Then, the task involves computing $B^*(y)$. The solution of FMP($B^*$) for CRI, TI, QI is $[1, 1, 1, 1, 1]$, $[1, 1, 1, 1, 1]$, $[0, 0, 0, 0.3, 0.3]$ respectively.

The problem for FMT is stated as follows. Let $A(x)$ denote "$x$ is small," $B(y)$ denotes "$y$ is large," and $B^*(y)$ denotes "$y$ is medium." Then, the task involves computing $A^*(x)$. The solution of FMT($A^*$) for CRI, TI, QI is $[0.3, 0.7, 1, 1, 1]$, $[0, 0, 0, 0, 0]$, $[0.3, 0.3, 0, 0, 0]$ respectively.

We can see that the CRI and TI solution for FMP is a trivial tautology, and the QI solution is much closer to the statement that "y is large." And CRI solution of FMT is very close to a trivial tautology. The TI solution represents a contradiction and this conclusion does not accord with human thinking. The QI solution is much closer to the statement that "x is small" (for Gödel and Łukasiewicz implications). Hence, QI method is considered to be more reasonable in the fuzzy reasoning process and it accords better with human thinking.

Some progress has been made in the formalization problem for fuzzy reasoning algorithms. Thus, Novak [5] gave a predicate formal representation of the solutions for the CRI method based on the fuzzy predicate logic BL. Pei [7] gave a predicate formal representation of solutions for the TI method and the strict logical proof. However, to the best of our knowledge, the logical foundation of the QI method has not been studied previously. Thus, in the present study, we consider the problem of formalizing the solutions for QI method based on the many-sorted first-order formal system of Monoidal t-norm based logic MTL$\forall_{ms}$ (because the fuzzy relation is characterized by the residuum and t-norm admit residuum if and only if it is left continuous; in addition, the many-sorted first-order formal system MTL$\forall_{ms}$ is an extension of the MTL, which is the logic of all left-continuous t-norms and their residuum). Moreover, we give the logical proof for solutions of the QI method based on the many-sorted first-order formal system MTL$\forall_{ms}$, which brings the QI method within the logical framework.

The remainder of this paper is organized as follows. Section 2 provides the basic definitions and notations used in this study. Section 3 presents the formal deductive systems for fuzzy reasoning. Section 4 gives the predicate formal representation of the solutions for the QI method based on the many-sorted first-order formal system MTL$\forall_{ms}$ and the strict logic proofs. We give our conclusions in Section 5.

## 2. Preliminaries

It is well known that a t-norm defines a residuum if and only if it is left-continuous (e.g., see [3]).

**Definition 2.1.** ([4]) A function $\otimes : [0, 1]^2 \to [0, 1]$ is called a t-norm if for all $a, b, c, \in [0, 1]$ that satisfy, $a \otimes 1 = a, a \otimes b = b \otimes a$, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ and $a \otimes c \leq b \otimes c$ whenever $a \leq b$.

**Definition 2.2.** ([2]) The residuated implication is defined by $R(a, b) = \sup\{x \in [0, 1] | T(a, x) \leq b\}$, $\forall a, b \in [0, 1]$, where $T$ is a t-norm on $[0, 1]$.

**QI Principle for FMP** ([12]). The solution $B^*$ of the FMP problem is the smallest fuzzy subset on $Y$ such that $(A(x) \to B(y)) \to ((A^*(x) \to A(x)) \to (A^*(x) \to B^*(y)))$ is a tautology, i.e., $B^*$ is the smallest fuzzy subset on $Y$ such that the following condition holds for every $x$ in $X$ and every $y$ in $Y$,