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Exchangeable choice functions ☆

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ABSTRACT

We investigate how to model exchangeability with choice functions. Exchangeability is a structural assessment on a sequence of uncertain variables. We show how such assessments constitute a special kind of indifference assessment, and how this idea leads to a counterpart of de Finetti's Representation Theorem, both in a finite and a countable context.

1. Introduction

In this paper, we study how to model exchangeability, a structural assessment for uncertainty models that is important for inference purposes, in the framework of choice functions, an interesting approach to modelling uncertainty. This work builds on earlier results by De Cooman et al. [7], De Cooman and Quaeghebeur [6] for sets of desirable gambles.

Choice functions are related to the fundamental problem in decision theory: how to make a choice from within a set of available options. In their book, von Neumann and Morgenstern [27] provide an axiomatisation of choice based on pairwise comparisons between options. Later on, many authors [2,15,22] generalised this idea and proposed a theory of choice functions based on choice between more than two elements. One of the aspects of Rubin's [15] theory is that, between any pair of options, the agent either prefers one of them or is indifferent between them, so two options are never incomparable. However, the agent may be undecided between two options without being indifferent between them; this will for instance typically be the case when there is no relevant information available. This is one of the motivations for a theory of imprecise probabilities [28], where incomparability and indifference are distinguished. Kadane et al. [12] and Seidenfeld et al. [19] generalise Rubin's [15] axioms to allow for incomparability.

Exchangeability is a structural assessment on a sequence of uncertain variables. Loosely speaking, making a judgement of exchangeability means that the order in which the variables are observed is considered irrelevant. This irrelevancy will be modelled through an indifference assessment. The first detailed study of exchangeability was given by de Finetti [8]; see Reference [9] for an overview of finite exchangeability for classical probability theory. We refer to the paper by De Cooman and Quaeghebeur [6, Section 1] for a brief historical overview.

In Section 2, we recall the necessary tools for modelling indifference with choice functions. Next, in Section 4, we derive de Finetti-like Representation Theorems for a finite sequence that is exchangeable. We take this one step further in Section 5, where we consider a countable sequence and derive a representation theorem for such sequences. In order to allow comparison with earlier work [6], we also provide representation theorems for sets of desirable gambles.

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2. Choice functions, desirability and indifference

Consider a real vector space $\mathcal{V}$, provided with the vector addition and scalar multiplication. Elements u of $\mathcal{V}$ are intended as abstract representations of *options* amongst which a subject can express his preferences, by specifying, as we will see below, choice functions. Often, options are bounded real-valued maps on the possibility space, interpreted as uncertain rewards, and therefore also called *gambles*. The set of all gambles on some domain $\mathcal{X}$ will be denoted by $\mathcal{L}(\mathcal{X})$. In this paper, we will rather focus on *vector-valued gambles*, because earlier work by Zaffalon and Miranda [30] has already shown that this leads to an approach to modelling uncertainty that is even more general than the typical imprecise probability approach. Moreover, as we have discussed in some detail in [26, Section 3], the account of coherent choice functions that Seidenfeld et al. [19] consider, can be embedded into our framework, under some mild conditions. To focus the ideas, let $\mathcal{X}$ be an arbitrary possibility space, and $\mathcal{R}$ be a finite set.[1] With a vector-valued gamble f on $\mathcal{X}$ we mean an element of the set $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ of gambles on the domain $\mathcal{X}\times\mathcal{R}$: indeed, for every x in $\mathcal{X}$, the partial map $f(x,\bullet)$ is a vector in $\mathbb{R}^{|\mathcal{R}|}$. We will commonly refer to $\mathcal{X}$ as the 'state part' of the domain, and to $\mathcal{R}$ as the 'rewards part'. Of course, by letting $|\mathcal{R}|=1$, we retrieve the set $\mathcal{L}(\mathcal{X})$ of (real-valued) gambles.

However, we will define choice functions on general real vector spaces $\mathcal{V}$ rather than on the more specific $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, because, as we will see later, we will need to define choice functions on *equivalence classes* of gambles, which are no longer gambles themselves, but still constitute a vector space.[2] Given any subset A of $\mathcal{V}$, we will define the *linear hull*

$$\operatorname{span}(A) := \left\{\sum_{k=1}^{n}\lambda_k u_k : n\in\mathbb{N}, \lambda_k\in\mathbb{R}, u_k\in A\right\}\subseteq\mathcal{V}$$

and the *positive hull*

$$\operatorname{posi}(A) := \left\{\sum_{k=1}^{n}\lambda_k u_k : n\in\mathbb{N}, \lambda_k\in\mathbb{R}_{>0}, u_k\in A\right\}\subseteq\operatorname{span}(A),$$

where $\mathbb{R}_{>0}$ is the set of all (strictly) positive real numbers. Furthermore, for any λ in $\mathbb{R}_{>0}$ and u in $\mathcal{V}$, we let $\lambda A+\{v\} := \{\lambda u+v : u\in A\}$. A subset A of $\mathcal{V}$ is called a *convex cone* if it is closed under positive finite linear combinations, i.e. if $\operatorname{posi}(A)=A$. A convex cone $\mathcal{K}$ is called *proper* if $\mathcal{K}\cap-\mathcal{K}=\{0\}$. With any proper convex cone $\mathcal{K}\subseteq\mathcal{V}$, we associate a vector ordering $\preceq_{\mathcal{K}}$ on $\mathcal{V}$ as follows: $u\preceq_{\mathcal{K}} v \Leftrightarrow v-u\in\mathcal{K}$ for any u and v in $\mathcal{V}$. For any u and v in $\mathcal{V}$, we write $u\prec_{\mathcal{K}} v$ if $u\preceq_{\mathcal{K}} v$ and $u\neq v$. We collect all the options u for which $0\prec_{\mathcal{K}} u$ in $\mathcal{V}_{\succ 0}$. When we work with vector-valued gambles, then $\mathcal{V}=\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and the ordering will be the standard one $\leq$, given by

$$f\leq g \Leftrightarrow (\forall x\in\mathcal{X}, r\in\mathcal{R})f(x,r)\leq g(x,r) \Leftrightarrow (\forall x\in\mathcal{X})f(x,\bullet)\leq g(x,\bullet)$$

We collect the positive gambles—gambles f for which $0<f$—in $\mathcal{L}(\mathcal{X}\times\mathcal{R})_{>0}$. Then $\leq$ corresponds to $\preceq_{\mathcal{K}}$ where we let $\mathcal{K} := \mathcal{L}(\mathcal{X}\times\mathcal{R})_{>0}\cup\{0\}$.

We denote by $\mathcal{Q}(\mathcal{V})$ the set of all non-empty *finite* subsets of $\mathcal{V}$. Elements of $\mathcal{Q}(\mathcal{V})$ are the option sets amongst which a subject can choose his preferred options.

A choice function C on $\mathcal{V}$ is a map $C\colon\mathcal{Q}\to\mathcal{Q}\cup\{\emptyset\}\colon A\mapsto C(A)$ such that $C(A)\subseteq A$. The idea underlying this definition is that a choice function C selects the set $C(A)$ of 'best' options in the *option set* A, or, on another interpretation, the ones that cannot be rejected. Our definition resembles the one commonly used in the literature [1,19,21], except for a restriction to *finite* option sets,[3] which, then again, is also not altogether unusual [10,16,20].

Not every such map represents rational beliefs; only the coherent ones are considered to do so.

Definition 1 *(Coherent choice function).* We call a choice function C on $\mathcal{V}$ *coherent*[4] if for all A, A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$, u and v in $\mathcal{V}$, and λ in $\mathbb{R}_{>0}$:

C1. $C(A)\neq\emptyset$;

[1] Mostly, $\mathcal{R}$ is interpreted as a set of 'rewards', but it need not have an interpretation. We can allow for countable $\mathcal{R}$ provided we then restrict ourselves to the linear space of those gambles f on $\mathcal{X}\times\mathcal{R}$ for which $\sum_{r\in\mathcal{R}} f(\bullet,r)$ is real-valued and bounded—a gamble on $\mathcal{X}$.

[2] This also allows us to connect our approach with the theory of coherent choice functions by Seidenfeld et al. [19], where the authors define their choice function on *horse lotteries* instead of gambles.

[3] The reason for our restriction to finite option sets is mainly a technical one: for instance the proof of Proposition 19 relies on the finiteness of the option sets. We refer to [23] for more information about this.

[4] Our rationality axioms are based on those by Seidenfeld et al. [19], slightly modified for use with sets of desirable options. Seidenfeld et al. [19] use horse lotteries as options, but, as mentioned before, by using vector-valued gambles their account of coherent choice functions can be embedded into ours; see our earlier work [26]. We would like to note here that Seidenfeld et al. [19] have a different definition of coherence: they call a choice function coherent if it is represented through E-admissibility by some set of (finitely-additive) real-valued probabilities and real-valued cardinal utilities. They introduce 4 axioms on choice functions that are equivalent (Seidenfeld et al. [19, Theorems 3 and 4]) to coherence, under some mild conditions.

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