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Concepts for decision making under severe uncertainty with partial ordinal and partial cardinal preferences ☆,☆☆

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ABSTRACT

We introduce three different approaches for decision making under uncertainty if (I) there is only partial (both cardinally and ordinally scaled) information on an agent's preferences and (II) the uncertainty about the states of nature is described by a credal set (or some other imprecise probabilistic model). Particularly, situation (I) is modeled by a pair of binary relations, one specifying the partial rank order of the alternatives and the other modeling partial information on the strength of preference. Our first approach relies on decision criteria constructing complete rankings of the available acts that are based on generalized expectation intervals. Subsequently, we introduce different concepts of global admissibility that construct partial orders between the available acts by comparing them all simultaneously. Finally, we define criteria induced by suitable binary relations on the set of acts and, therefore, can be understood as concepts of local admissibility. For certain criteria, we provide linear programming based algorithms for checking optimality/admissibility of acts. Additionally, the paper includes a discussion of a prototypical situation by means of a toy example.

1. Introduction

One of the constantly recurring topics discussed in the community of researchers working with imprecise probabilities (and on ISIPTA conferences in particular) is defining meaningful criteria for decision making under complex uncertainty, finding persuading axiomatic justifications for these criteria and providing efficient algorithms capable to deal with them. Examples for such works are ranging from rather early IJAR and ISIPTA contributions by, e.g., [23,1,48,53] to more recent ones by, e.g., [55,58,24,36,4].

However, in the vast majority of works in this field, the complexity underlying the decision situation is assumed to solely arise from the fact that the decision maker's beliefs on the mechanism generating the states of nature are expressed by an imprecise probabilistic model. In contrast, the cardinal utility function adequately describing the decision maker's preference

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structure is often unquestioned and assumed to be precisely given in advance.[1] Unfortunately, also this can be problematic. Wrongfully pretending to have perfect information on the level of utilities might lead to bad decision making just as doing the same on the level of beliefs: What's worth a decision that is derived on the basis of an inadequate utility function?

For this reason, our paper generalizes both the classical setting of decision making under risk as well as the generalized setting of decision making under ambiguity to situations in which the assumption of a known cardinal utility structure is no longer justified. Particularly, we consider the case that the (information on the) decision maker's preference structure is both of partially ordinal and of partially cardinal scale and, therefore, no longer can be characterized by (a set of positive linear transformations of) *one* cardinal utility function. Instead, we model the decision maker's utility by the set of all utility representations that are compatible with both the ordinal and the cardinal information concerning her preferences.

The paper is structured as follows: In Section 2, we give a brief overview on the background of our work and show how our approach naturally fits into this picture. Moreover, we discuss related literature and the connections to our work. In Section 3, we introduce the crucial concept of a *preference system* over a set of alternatives that allows for modeling partially ordinal and partially cardinal preference structures. Section 4 introduces three different approaches for decision making with acts taking values in a preference system by proposing decision criteria based on generalized expectation intervals (Section 4.2), on global comparisons of acts (Section 4.3) and on pairwise comparisons of acts (Section 4.4). For certain criteria, we give linear programming driven algorithms for checking feasibility of acts in finite decision settings. Section 5 is devoted to an application of the theory. There, we illustrate all the concepts developed in the paper in an example and thereby also show a class of situations in which our approach seems natural: The case where the consequences that acts can attain belong to some product space with both ordinal and cardinal dimensions. Section 6 concludes the paper.

2. Fundamentals underlying our approach and related literature

In classical subjective expected utility theory (SEUT), the decision maker (synonymously called agent in the following) is assumed to be able to specify (I) a real-valued *cardinal* utility function u (unique up to a positive linear transformation) representing her preferences on a set A of alternatives and (II) a unique and *precise* subjective probability measure π on the space S of states of nature adequately specifying her beliefs on the occurrence of the different states $s \in S$. Once these two ingredients are specified, according to SEUT, the decision maker should choose any act $X : S \to A$ that maximizes the expected utility $\mathbb{E}_\pi(u \circ X)$ with respect to her utility function u and her subjective probability measure π among all other available acts.

However, as is well known, in practice both assumptions (I) and (II) often turn out to be systematically too restrictive. In particular, (I) demands the decision maker to act in accordance with the axioms of von Neumann and Morgenstern, i.e. to be able to specify a complete preference ranking of all simple lotteries on the set A that is both independent and continuous (see, e.g., [17, Ch. 8] for details), whereas (II) requires that the decision maker can completely order the resulting utility-valued acts by preference in accordance with the axioms of de Finetti, i.e. continuous, additive and monotone (see, e.g., [19, Ch. 9] for details).

Consequently, there exists plenty of literature relaxing these assumptions. If *only* (II) is violated in the sense that there is only *partial* probabilistic information on the occurrence of the states of nature together with a perfectly cardinal preference structure (represented by a cardinal utility function u), the common relaxation is to allow for *imprecise* probabilistic models for representing the probabilistic information (for instance one could use the credal set $\mathcal{M}$ of all probability measures that are compatible with the given probability constraints). In this case, one can define optimality of acts, for instance depending on the attitude of the decision maker towards the ambiguity underlying the situation, in terms of some imprecise decision criterion such as:

- Γ-*maximin* (Γ-*maximax*): Choose any arbitrary act X yielding maximal expected utility with respect to the worst (best) compatible probability measure, i.e. that maximizes the value $\inf_{\pi \in \mathcal{M}} \mathbb{E}_\pi(u \circ X)$ (the value $\sup_{\pi \in \mathcal{M}} \mathbb{E}_\pi(u \circ X)$) among all available acts.
- *Maximality*: Dismiss each act X for which there is available another act Y that dominates it in expectation with respect to all compatible probability measures, i.e. for which it holds that $\mathbb{E}_\pi(u \circ X) < \mathbb{E}_\pi(u \circ Y)$ for all $\pi \in \mathcal{M}$.
- *E-admissibility*: Dismiss each act X that does not maximize expected utility $\mathbb{E}_\pi(u \circ X)$ among the available acts with respect to at least one compatible probability measure $\pi \in \mathcal{M}$, i.e. where for all $\pi \in \mathcal{M}$ there exists an act Y_π with $\mathbb{E}_\pi(u \circ X) < \mathbb{E}_\pi(u \circ Y_\pi)$.

The original sources of the criteria just discussed are given in [30,34,35,20,60]. Further criteria for the case of cardinal utility and imprecise probabilities, each in its own way taking into account the whole set $\mathcal{M}$ of compatible probability measures, are reviewed in, e.g., [22]. Additionally, there exists a variety of efficient and powerful algorithms to deal with this kind of violation of the classical assumptions (see, e.g., [57,28,21,25]). However, note that the assumption of a cardinal utility

[1] Exceptions include Montes [39, Section 4.2.1], who uses set-valued utility functions, Landes [32] who axiomatically characterizes preferences over utility intervals and Troffaes and Sahlin [56], who propose elicitation procedures for partially specified utility functions. These references, among others, are discussed in some more detail at the end of Section 2.