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Fuzzy approximations of fuzzy relational structures

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ABSTRACT

In social network analysis, relational structures play a crucial role in the processing of complex data. This paper proposes a fuzzy relational structure which consists of a non-empty universal set and a set of fuzzy relations of finite arity. In order to deduce knowledge hidden in a fuzzy relational structure we present the concept of bisimulations as indiscernibility with respect to the fuzzy relational structure. Furthermore, we give a method of computing the largest bisimulations over fuzzy relational structures and present computational examples of the method. Finally, the fuzzy rough set analysis of fuzzy relational structures is discussed.

1. Introduction

The theory of rough sets was firstly proposed by Pawlak in 1982 [31–33]. It is a useful mathematical tool for dealing with attribute data. In this theory, attribute data can be represented by a data table which comprises a set of objects and a set of attributes. The objects of a given data table can be identified within the limits of imprecise data by an indiscernibility relation induced by different values of attributes characterizing these objects. In other words, the indiscernibility relation induced by different values of attributes enables us to characterize a set of objects by a pair of sets, called the lower and upper approximation of the set of objects. Although databases only contain attribute information about objects in many practical cases, data about the relationships between objects has become increasingly important in decision analysis in recent years [17]. A remarkable example can be found in social network analysis, where the principal types of data are attribute data and relational data [40]. Liau et al. [23] studied granulation based on relational information between objects from the viewpoint of modal logic. More recently, Fan [17] further investigated relational information systems and extended rough set analysis from attribute information systems to relational structures which consists of a non-empty universal set and a set of relations. In this paper we pay attention to the generalization of relational structures in fuzzy environment and present bisimulations as indiscernibility with respect to fuzzy relational structures.

Milner [25] and Park [34] introduced bisimulation as a concept that provides an effective method for reducing the complexity of concurrent processes and studying the equivalence of automata. In particular, bisimulation is exploited to reduce the state space of a system by combining bisimilar states. In the last thirty years, it has been applied to computer science, modal logic and set theory [16,19,24,26,27,37,41,14,18]. Recently, bisimulations have been introduced to fuzzy systems by

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two general approaches. One is based on a binary relation on the state space of a fuzzy system such that related states have exactly the same possibility degree of making a transition into every class of related states [36,2,42,5,6,48,7,15]. The other is based on a fuzzy relation on the state space. Ćirić and his colleagues [8,21,9,10,39,11] proposed two types of simulations and four types of bisimulations for fuzzy automata.

The purpose of the paper is to discuss the fuzzy rough set analysis of fuzzy relational structures by bisimulations which are similar to forward bisimulations considered in [10].

The rest of this paper is structured as follows. In Section 2 we introduce necessary terminology and notions of residuated lattices, fuzzy sets, and n-ary fuzzy relations and extend the concept of composition from binary fuzzy relations to n-ary fuzzy relations. Sections 3 presents a fuzzy relational structure which consists of a non-empty universal set and a set of fuzzy relations. In Section 4 we give main results on the computing of the largest bisimulations over fuzzy relational structures. Examples presented in Section 5 demonstrate the application of our method. Section 6 concludes this work.

2. Preliminaries

For the terminology and basic notions in this section we refer to [10,3,4,22].

A residuated lattice is an algebra $\mathscr{L}=(L,\wedge,\vee,\otimes,\rightarrow,0,1)$ such that

(1) $(L,\wedge,\vee,0,1)$ is a lattice with the least element 0 and the greatest element 1,
(2) $(L,\otimes,1)$ is a commutative monoid with the unit 1,
(3) $\otimes$ and $\rightarrow$ form an adjoint pair, i.e., they satisfy the adjunction property: for all $a,b,c\in L$,

$$a\otimes b\le c \Leftrightarrow a\le b\rightarrow c.$$

In addition, a residuated lattice is called complete if $(L,\wedge,\vee,0,1)$ is a complete lattice. If any finitely generated subalgebra of residuated lattice $\mathscr{L}$ is finite, then $\mathscr{L}$ is called locally finite.

For any family $a_i, i\in I$, of elements of L, we write $\bigvee_{i\in I} a_i$ for the supremum of $\{a_i\}_{i\in I}$, and $\bigwedge_{i\in I} a_i$ for the infimum. An operation $\leftrightarrow$ is defined by $a\leftrightarrow b=(a\rightarrow b)\wedge(b\rightarrow a)$. It can be verified that $\otimes$ is isotonic in both arguments and for all $a,b,c\in L$ and $\{a_i\}_{i\in I},\{b_i\}_{i\in I}\subseteq L$ the following hold:

$$(\bigvee_{i\in I} a_i)\otimes a=\bigvee_{i\in I}(a_i\otimes a), \tag{1}$$

$$(a\leftrightarrow a)=1,\ (a\leftrightarrow b)=(b\leftrightarrow a),\ (a\leftrightarrow b)\otimes(b\leftrightarrow c)\le(a\leftrightarrow c), \tag{2}$$

$$a\leftrightarrow b\le a\otimes c\leftrightarrow b\otimes c, \tag{3}$$

$$\bigwedge_{i\in I}(a_i\leftrightarrow b_i)\le(\bigvee_{i\in I} a_i)\leftrightarrow(\bigvee_{i\in I} b_i). \tag{4}$$

Let L be the real unit interval $[0,1]$, $a\wedge b=min\{a,b\}$ and $a\vee b=max\{a,b\}$. Suppose that $a\otimes b=min\{a,b\}$ and $a\rightarrow b=1$ if $a\le b$, and $=b$ otherwise. Then the structure of a residuated lattice with assumptions above is called Gödel structure. The structure where $a\otimes b=a\cdot b$ and $a\rightarrow b=1$ if $a\le b$, and $=b/a$ otherwise is the Goguen (product) structure.

From now on, $\mathscr{L}=(L,\wedge,\vee,\otimes,\rightarrow,0,1)$ will be a complete residuated lattice. Let U be a non-empty set. A fuzzy set A in U is a mapping $A:U\rightarrow L$. The set U is called a universe. $A(a)$ is called the degree of membership of a in A. The set of all fuzzy sets in U will be denoted by L^U. The support of a fuzzy set A is a set defined as $supp(A)=\{a\in U: A(a)>0\}$. If $supp(A)=\{a_1,\cdots,a_n\}$ is a finite set, then we often use Zadeh's notation

$$A=\frac{A(a_1)}{a_1}+\frac{A(a_2)}{a_2}+\cdots+\frac{A(a_n)}{a_n}$$

where the term $\frac{A(a_i)}{a_i}$ and $i=1,\cdots,n$, signifies that $A(a_i)$ is the degree of membership of a_i in A and the plus sign represents the union.

Let A, B and $A_i\in L^U(i\in I)$. A and B are said to be equal, which is denoted by $A=B$, if $A(a)=B(a)$ for all $a\in U$. The inclusion of A and B is defined by $A\le B$ if $A(a)\le B(a)$ for all $a\in U$. Further, given this partial order the meet $\bigwedge_{i\in I} A_i$ of $\{A_i\}_{i\in I}$ is defined as $(\bigwedge_{i\in I} A_i)(a)=\bigwedge_{i\in I} A_i(a)$, for all $a\in U$, the join $\bigvee_{i\in I} A_i$ of $\{A_i\}_{i\in I}$ is defined as $(\bigvee_{i\in I} A_i)(a)=\bigvee_{i\in I} A_i(a)$, for all $a\in U$, and the product $A\otimes B$ is a fuzzy set defined by $(A\otimes B)(a)=A(a)\otimes B(a)$, for all $a\in U$.

A fuzzy set in $U_1\times\cdots\times U_n$ is called an n-ary fuzzy relation between $U_1,U_2,\cdots$, and U_n. For simplicity, we call an n-ary fuzzy relation as a fuzzy relation. If $U_1=\cdots=U_n=U$, we speak of n-ary fuzzy relation on a set U. The set of all n-ary fuzzy relations on U will be denoted by L^{U^n}.

For $\varphi,\psi\in L^{U^2}$, the composition $\varphi\circ\psi$ is a fuzzy relation from L^{U^2} defined by

$$(\varphi\circ\psi)(a,c)=\bigvee_{b\in U}\varphi(a,b)\otimes\psi(b,c),$$

for all $(a,c)\in U^2$.

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