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Explicit analytical formulae of ranking indices without the requirement of multiplicative compatibility

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A R T I C L E I N F O

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The ranking of fuzzy numbers is often given by a special class of utility functions, better known as ranking indices of the first kind. Recently, we have proved that the effective ranking of fuzzy numbers from a set containing all real numbers can be done by involving a very particular class of ranking indices. In this way, the searching of effective ranking methods is essentially simplified. In this paper, firstly we extend this result in a sense which allows us to find explicit formulae for ranking indices defined on other important subsets of fuzzy numbers, as for example the set of positive trapezoidal fuzzy numbers or the set of positive triangular fuzzy numbers. Another novelty is that, reducing all those ranking indices that generate the same ordering to a special member of their class, under some reasonable regularity assumptions, we find all the orderings on trapezoidal fuzzy numbers, or positive trapezoidal fuzzy numbers, respectively, which satisfy all the basic requirements of Wang and Kerre, except for the compatibility with scalar multiplication. This special member of the class is what we call a basic ranking index, that is, a ranking index which assigns to a fuzzy number a real number that belongs to its support. Actually, these basic ranking indices can be extended to linear functionals, so their form is very simple and ready to be used in applications. Similar results are obtained for the set of triangular fuzzy numbers and positive triangular fuzzy numbers, respectively. The key element in all these theoretical results, is that under some reasonable regularity properties, we can reduce a ranking index to this so called basic ranking index, which has a simpler form but generates the same order, that is, a binary relation with the same graph. As a first application, we review several ranking approaches got from ranking indices which are additive functions with respect to addition of fuzzy numbers and check their properties from the list of Wang and Kerre. Then, we do the same for a group of ranking indices none of them being additive functions with respect to the addition of fuzzy numbers. We can always do that, as long as we reduce a nonadditive ranking index to a basic ranking index, with both ranking indices generating the same ordering. For the newly obtained ranking index, we will just need to study additivity or scale invariance, depending on which type of property from the list of Wang and Kerre are we interested in. An interesting special case of nonadditive ranking index that cannot be reduced to a basic ranking index is the so called signed distance. We slightly modify this ranking index, so that it can be reduced to a basic ranking index. In this way, we never get contradictory results for fuzzy numbers with disjoint supports. Lastly, we propose a method to extend the ordering of positive trapezoidal fuzzy numbers to the ordering of arbitrary positive fuzzy numbers.

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1. Introduction

Ranking of fuzzy numbers is one of the important issues in fuzzy sets theory since there are so many research areas (economy, decision theory, approximate reasoning and others) which rely on the ranking of fuzzy numbers. Unfortunately, it seems that it is hard to find a ranking approach effective in any environment, taking into account that experimental studies always pointed out so called counterintuitive results. The aim of the present paper is to complete the results obtained in [9–11], by finding important classes of orderings defined on some special sets of fuzzy numbers and such that a number of reasonable criteria are satisfied by these orderings. These criteria are derived from the list of Wang and Kerre (see [33]) and they are accepted by all researchers that work in this topic. What is more, they should stay at the basis of any research in the ranking of fuzzy numbers based on binary relations. We discuss the case of ranking approaches obtained via binary relations generated by so called ranking indices of the first kind, that is, real functions defined on the set of interest. It means that fuzzy numbers are ranked by using the natural order between their real images. In economy such functions are often referred as utility functions. Note that there are more complex constructions by using so called ranking indices of the second kind (see [15], [33]). These ranking indices of the second kind generate binary relations on any finite subset of the given set. But these binary relations are not necessarily related in the sense of being covered by a single binary relation defined on the given set of fuzzy numbers. If such a single binary relation would exist, then ranking indices of the second kind would be actually ranking indices of the first kind. Although we do not discuss them here, it seems that our results could open the premises to study ranking indices of the second kind too. More details on this issue are given in the conclusions section. It is important to note that ranking indices are not the only tool used to rank fuzzy numbers. For example, in [24] four ranking methods are proposed in the context of Possibility Theory. Then, in [21] formal relations between some existing rankings from the literature and some stochastic orderings, in the context of imprecise probability theory are studied.

There are hundreds of papers where fuzzy numbers are ranked using ranking indices of the first kind (from now on we refer to them just as ranking indices). Interestingly, the methodology is quite the same in the last 20 years. Either we look on older papers (see, e.g., [16], [17], [28]), or on some recent ones (see, e.g., [2–6], [26], [32], [35], [38,39]), the authors always try to convince us that their method is better comparing to others, because it gives better interpretations in some of the so called (by Wang and Kerre) "controversial cases". As we have already indicated in [10], this approach to the ranking problem has a dose of subjectivity because from a few examples we cannot decide which approach is better in general. What is common to these papers and many others as well, is that authors try to validate the ranking procedure by showing that it satisfies as many as possible properties from the list of basic requirements on ordering procedures given by Wang and Kerre in their famous paper [33]. It is a list of reasonable properties like reflexivity, transitivity, compatibility with addition, compatibility with multiplication, and last but not least, a group of two properties which always allow to compare fuzzy numbers having supports that intersect in at most one single point. From these last two requirements, in addition, we get that the natural order between real numbers is preserved. Another common feature in these papers is that all examples apply almost exclusively on trapezoidal or triangular fuzzy numbers. These sets, besides their obvious suitability in concrete applications, they also possess some nice regularity properties, like being closed under addition or scalar multiplication.

All the above facts inspired us to propose a more rigorous, axiomatic study of the ranking problem. Our greatest ambition is to give positive answers to some pertinent questions. In what follows, two ranking indices that generate the same ordering on some set $\mathcal{S}$ will be called equivalent ranking indices. Considering a class $\mathcal{S}$ of fuzzy numbers, is it possible to find all ranking indices which generate orderings on $\mathcal{S}$ that satisfy all the basic requirements of Wang and Kerre? When we say all ranking indices, we mean that from a class of equivalent ranking indices a single representant is enough since all these ranking indices will generate the same ordering. Then, can we find all the ranking indices which generate orderings on $\mathcal{S}$ that satisfy only a part of these requirements? More specifically, abstraction making of equivalent ranking indices, can we find all ranking indices that generate orderings which do not necessarily satisfy compatibility with scalar multiplication (that is, order is not necessarily preserved after multiplication of fuzzy numbers with a real number) but satisfy all the other requirements of Wang and Kerre? And what about those ranking indices which generate orderings for which we do not need compatibility with algebraic operations? Considering the first question from above, the problem is completely solved when $\mathcal{S}$ is the set of trapezoidal fuzzy numbers, respectively the set of triangular fuzzy numbers and, when we are interested in finding the class of ranking indices generating orders that satisfy all the criteria of Wang and Kerre (see papers [9,10]). In this very contribution, our aim is to give answers for the remaining questions presented just above. We must insist on the fact that our work is really objective, that is, we do not intend to propose any kind of hierarchy between the ranking indices. We leave this part to practitioners. What we can do, is to provide all possible ranking indices generating orders that satisfy a number of reasonable criteria from the list of Wang and Kerre. So, we can say that they are more or less effective with respect to these criteria. To sum up, instead of proposing a particular ranking method (the literature is so rich with such approaches), hoping that it behaves well in respect to the requirements from the list of Wang and Kerre, we find analytical formulae for all ranking indices (abstraction making of equivalent ranking indices) that generate orders satisfying all or just a part of these criteria. From this perspective, we should include our contribution in the list of papers where the ranking problem is treated in an objective way, such as for example, papers [13], [14], [33], or the earlier mentioned paper [21]. In [13] and [14], respectively, orderings obtained from existing ranking indices are reviewed based on a numerical analysis, but not in respect to the criteria of Wang and Kerre. In [33], where these criteria are introduced, a group of ranking indices are examined in respect to these criteria. Therefore, this paper is the most related one to our