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Nguyen type theorem for extension principle based on a joint possibility distribution

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ABSTRACT

In this paper, first we prove that making abstraction of the output obtained from the interactive extension principle based on a joint possibility distribution, in the case of unimodal fuzzy numbers and when the function that generates the operation is continuous and strictly increasing in each argument restricted to the support of each fuzzy number used in the process, then we can use joint possibility distributions with the property that the left/right side of the output is obtained from the convolution of the values in the left/right side of these fuzzy numbers. Then, considering joint possibility distributions with the aforementioned property, we find an Nguyen type characterization of the level sets of the output based on interactive extension principle, in terms of the level sets of the fuzzy numbers used in the process. These two key results complete well-known results obtained in the case of Zadeh's extension principle and also in the case of triangular norm-based extension principle. As an interesting corollary, in the special case of unimodal fuzzy numbers, the Nguyen theorem can be used to present a new proof concerning necessary and sufficient conditions on the equality of the outputs based on joint possibility distributions, respectively based on Zadeh's extension principle.

1. Introduction

One of the most important challenges in the theory of fuzzy sets is how to operate with fuzzy quantities. The most natural way to do it is to use Zadeh's famous extension principle. This method is based on a convolution process employing the minimum operator. Therefore, replacing this operator by an arbitrary triangular norm one obtains a generalization of the extension principle called triangular norm-based extension principle. Numerous scholars addressed this topic (see, e.g., [16], [19], [22–27], [29–35]). Especially additions and multiplications were investigated but the general framework is also covered and nevertheless important. A further generalization of standard operations based on the extension principle is possible by using so called joint possibility distributions introduced by Fullér and Majlender in 2004 (see [20]). They called this approach as the interactive extension principle. A joint possibility distribution can be regarded as the analogue in possibility theory of a joint probability distribution. This time the marginal distributions are fuzzy numbers. In paper [8] the authors proposed a so called interactive extension principle which is based on a convolution method employing joint possibility distributions. This method is even more general (for a fixed sample of fuzzy numbers) than the one based

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on triangular norms. Joint possibility distributions and their properties have been investigated in the last decade (see, e.g., [3], [7–15]). Recently, joint possibility distributions were used in the sudy of initial value problems (see [18]) and multi-period portofolio optimization (see [36,37]). Recently, we studied under which conditions operations based on the extension principle coincide with operations based on the interactive extension principle (see [11], [13], [14]). In this paper, we are interested in representing the level sets of the outputs based on interactive extension principle, in terms of the level sets of fuzzy numbers used in the process. In the case of standard extension principle such a result was proved in the famous paper of Nguyen (see [28]). Then, Fullér and Keresztfalvi generalized Nguyen's result (see [19]) as they characterized the level sets for outputs obtained from the triangular norm-based extension principle. They considered only the bidimensional case which was then extended for the n-dimensional case by Wu in paper [35]. Most notably, in Wu's paper even more general operators are proposed than triangular norms. From this discussion we observe that a characterization of the level sets of outputs based on interactive extension principle would be really important in order to complete the previous studies. We will obtain this characterization in the present paper. It is important to mention that in papers [19] [28], and [35] respectively, it suffices to assume that the function that generates the algebraic operation is continuous. This is so mostly because the triangular norms are symmetric operators. We cannot use such property in case of extensions based on joint possibility distributions. This is why we will need to compute the values of a joint possibility distributions at the endpoints of the level sets for each fuzzy number used in the process. For this reason, our results can be applied for functions that are strictly increasing in each argument restricted to the support of the corresponding fuzzy number. A particular case of such function is addition viewed as a bivariate operator. A detailed study of this particular case can be found in paper [12]. Another important difference in respect to the previous methods is that the output based on the interactive extension principle does not preserve the core of the output based on the standard extension principle. This is why we will insist with the case of unimodal fuzzy numbers (still a very important class of fuzzy numbers) since in this case we have preservation of the core. Lastly, we mention that the triangular norm-based extensions where the function that generates the operation satisfies the before mentioned monotonicity requirement, have the very nice property that the left side of the fuzzy number is obtained from the convolution of the values that are in left side of the fuzzy numbers and similarly for the right side. Even if in general this is not true for extensions based on joint possibility distributions, in the case of unimodal fuzzy numbers we will prove that if we make abstraction on the final output, we can use only joint possibility distributions such that the left/right side of the output is obtained by the convolution of values from the left/right side of each fuzzy number.

The paper is organized as follows. Section 2 contains the basics about fuzzy numbers, triangular norms and joint possibility distributions. Section 3 presents the three types of extension principle discussed in this paper, that is, Zadeh's extension principle, the extension principle based on triangular norms and the extension principle based on joint possibility distributions. Section 4 presents examples of joint possibility distributions. Some of them are well-known in the literature but we also propose joint possibility distributions that will be useful examples to explain the differences between triangular norm-based extensions and extensions based on joint possibility distributions. Section 5 presents the first key result of the paper. We prove that in the case of unimodal fuzzy numbers and when the function that generates the operation is continuous and strictly increasing in each argument restricted to the support of the corresponding fuzzy number, if we make abstraction of the output, we can use only joint possibility distributions such that the left/right side of the output is obtained from the convolution of the values from the left/right side of each fuzzy number used in the process. As an important corollary, we will use the main result for the special case when we deal with interactive addition of fuzzy numbers. Section 6 contains the next key result where we find a characterization of the level sets of the output based on interactive extension principle, in terms of the level sets of the fuzzy numbers used in the process. This result completes the results of Nguyen, Fullér and Keresztfalvi, and the one of Wu, discussed earlier. The result is quite general when we are under the same hypotheses as in the previous key result. Again, we apply the main result for the special case of interactive addition. Then, as an interesting corollary, in the special case of unimodal fuzzy numbers, we present a new proof of a result in paper [13], where necessary and sufficient conditions are proposed for the equality of the outputs based on joint possibility distributions, respectively based on Zadeh's extension principle. The paper ends with conclusions summarizing the main results and addressing further research in this topic.

2. Preliminaries

This section describes the basic concepts used in the paper. To make the paper more readable, we leave to the next section the discussion on the concrete problem addressed in this work.

A fuzzy number A is characterized by a membership function $A:\mathbb{R}\to[0,1]$ which is normal, quasi-concave (also referred as fuzzy convex), upper semicontinuous and with bounded support in topological sense considering the usual topology on $\mathbb{R}$. For some $\gamma\in(0,1]$, the so called γ-cut of u is denoted with A^γ, where

$$A^\gamma=\{x\in\mathbb{R}:A(x)\geq\gamma\}.$$

For $\gamma=1$ we obtain the core of A, denoted $\mathrm{core}(A)$, hence $\mathrm{core}(A)=A^1$. The support of A will be denoted with $\mathrm{supp}(A)$ and it can also be regarded as the 0-cut of A, where

$$A^0=\mathrm{cl}\{x\in\mathbb{R}:A(x)>0\},\tag{1}$$

and here "cl" denotes the closure operator considering the usual topology on $\mathbb{R}$. Thus, we have $\mathrm{supp}(A)=u^0$.