International Journal of Approximate Reasoning 95 (2018) 36–45

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Inconsistency of special cases of pairwise comparisons matrices

V. Čerňanová [a], W.W. Koczkodaj [b,*], J. Szybowski [c]

[a] Department of Mathematics and Computer Science, Faculty of Education, Trnava University, Priemyselná 4, 918 43 Trnava, Slovakia
[b] Computer Science, Laurentian University, Sudbury, Ontario P3E 2C6, Canada
[c] AGH University of Science and Technology, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Kraków, Poland

ARTICLE INFO

Article history:
Received 30 December 2017
Received in revised form 23 January 2018
Accepted 24 January 2018
Available online 31 January 2018

Keywords:
Pairwise comparisons
Inconsistency indicator
Eigenvalue
Toeplitz matrix

ABSTRACT

This study presents special cases of inconsistent pairwise comparisons PC matrices. The analysis of the selected inconsistency indicators is provided. One of the compared inconsistencies is the popular eigenvalue-based consistency index (CI) which fails to follow axiomatization recently published by this journal. The other inconsistency, based on the exponentially invertible measure (also known as Koczkodaj's inconsistency indicator or Kii) follows the axiomatization.

All studied special cases of PC matrices are Toeplitz matrices with only three different entries 1, x, and $1/x$. A circulant matrix has been used for pairwise comparisons to analyze inconsistency. Although this class of PC matrices may be perceived as restricted, it is general enough to cover all values of the eigenvalue-based inconsistency index from the lowest to the highest. Both exact mathematical expressions and estimations, where the exact expression was impossible to find, are provided.

1. Introduction

The first documented application of pairwise comparisons has been recently attributed to the works of Ramon Llull, the 12-th century mathematician, logician, philosopher, Majorcan writer, and mystic [4]. However, it is easy to envision that the practical use of pairwise comparisons could have taken place in the Stone Age. Evidently, the need existed for stones to be compared to each other in pairs to decide which is better suited to use as a tool. When we have no unit of measure (e.g., for software quality), we may consider the construction of a pairwise comparisons matrix (PC matrix) to express our assessments based on relative comparisons of attributes (such as software safety or reliability). In this work, we examine inconsistency in some special cases of PC matrices. It has been discovered (by the first author) that they belong to the class of well-known Toeplitz matrices.

After all, common sense and an old adage (commonly attributed to Creighton Abrams) calls for "take one bite at a time" when it comes to eating an elephant. Our elephant is the processing of subjective data, especially for the decision making where the "satisfying" approach is often used. Herbert A. Simon, the Nobel prize winner, proposed bounded rationality ("satisfying") as a vital alternative to the exclusiveness of using mathematical theory for decision making. Pairwise comparisons supports the concept of bounded rationality. Pairwise comparisons matrices are $n \times n$ reciprocal matrices $\mathbf{A} = (a_{ij})$ with positive entries. A PC matrix $\mathbf{A}$ is called *reciprocal* if $a_{ij} = 1/a_{ji}$ for $i, j = 1, \ldots, n$. This implies that $a_{ii} = 1$ for all i;

* Corresponding author.
E-mail addresses: vieracernanova@hotmail.com (V. Čerňanová), wkoczkodaj@cs.laurentian.ca (W.W. Koczkodaj), szybowsk@agh.edu.pl (J. Szybowski).

https://doi.org/10.1016/j.ijar.2018.01.004
0888-613X/

hence the matrix trace is $Tr(\mathbf{A}) = n$. However, it is worth noting that blind wine testing may result not only in the lack of reciprocity but even the lack of 1s on the main diagonal as comparing the same wine to itself may not always give 1. Due to the Internet, different assessors may compare software project components in different locations. In such situations, it is even anticipated that some assessments may not be reciprocal.

The paper is organized as follows. In Section 2, some definitions and properties, relating to matrix methods are outlined together with some basic concepts of pairwise comparisons. Sections 3 to 7 are devoted to various special types of pairwise comparisons matrices, which are $n \times n$ Toeplitz matrices with entries 1, x, $1/x$. In most cases a closed form of the consistency index (4) is found as a function of x and n.

2. Preliminaries

A pairwise comparisons matrix **A** is called *consistent* (or *transitive*) if

$$a_{ij} \cdot a_{jk} = a_{ik} \text{ for } i, j, k = 1, \ldots, n. \tag{1}$$

This is known as the consistency condition. While every consistent matrix is reciprocal, the converse is false in general. If the consistency condition does not hold, the matrix is inconsistent (or intransitive). Given a reciprocal $n \times n$ matrix **A** which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix $\mathbf{A}'$ which is "as close as possible" to the given matrix. Inconsistency was analyzed in [7,10,6] if not earlier. Axiomatization for inconsistency indicators was recently proposed in [17]. Reasoning for the normalization (which has become one of the axioms) was provided in [14].

The matrix entries a_{ij} express a relative preference of an entity E_i over E_j often by using a rating scale: "E_i is x times more essential than E_j". An entity could be any object, attribute of it or a stimulus. The scale was mathematically analyzed in [5]. Consistent matrices correspond to the ideal situation in which there are exact values $E_1, \ldots, E_n$ for the stimuli, since condition (1) is automatically satisfied when $a_{ij} = E_i/E_j$ for all (even random) positives values E_i. This is an important observation, since the implication of it is that, a problem of approximation is really a problem of selecting a norm and distance minimization. Notice that the vector $(E_1, \ldots, E_n)$ is unique up to a multiplicative constant. For the Euclidean norm the vector of geometric means (which is equal to the principal eigenvector for a consistent PC matrix) is the one which generates the hierarchy of the entities. The seminal study [19] had a considerable impact on the pairwise comparisons research. However, it has strongly endorsed the use of the eigenvector, corresponding to the principal eigenvalue, for approximation of a given inconsistent but reciprocal PC matrix. Numerous studies show the lack of evidence for the superiority of the eigenvector solution. It is expressed in the highly cited [2] and a sizable collaboration (recently published) [13].

In layman's terms, when we compare in pairs three entities: E_1, E_2, and E_3, $(E_1/E_2) \times (E_2/E_3)$ is expected to yield the same result as the comparison E_1/E_3. In practice, however, relation (1) does not hold when these three comparisons are carried independently. When the comparisons are subjective (e.g., more important, safer, etc.) or entities are subjective (safety, reliability, etc.), the inconsistency between the above three comparisons may be substantial. The lack of inconsistency may be even suspicious since data may be "doctored" but the high level of inconsistency cannot be regarded as desirable or even tolerated. The common sense rule "GIGO" (GIGO stands for "garbage-in, garbage-out") in computer science and information technology stresses the fact that output quality depends on the quality of input data. Certainly, exceptions exist.

The following observations are known properties of consistent matrices (some of them are introduced in [19]), and trivial to check:

Observation 2.1.

(i) *Each row of a consistent matrix is a constant multiple of any other row. Similarly, each column of a consistent matrix is a constant multiple of any other column.*
(ii) 0 *is an eigenvalue with multiplicity $n-1$ of any consistent $n \times n$ matrix, hence the unique non-zero eigenvalue of a consistent $n \times n$ matrix is n.*
(iii) *Every column of a consistent PC matrix is its eigenvector corresponding to the eigenvalue n.*

In [1], the greatest lower bound and the least upper bound were found for the Perron root (called principal or Perron's eigenvalue, and equal to the spectral radius of the matrix with non-negative entries) of a PC matrix:

Theorem [quoted "as is" from [1]]: Let **A** be a positive reciprocal matrix with entries $1/x \le a_{ij} \le x, 1 \le i, j \le n$ for some $x \ge 1$, and let $\lambda_{\max}$ denote the largest eigenvalue of **A** in modulus, which is known to be real and positive from the Perron–Frobenius theorem. Then

$$n \le \lambda_{\max} \le 1 + \frac{1}{2}(n-1)\left(x + \frac{1}{x}\right), \tag{2}$$

the lower and upper bound being reached if and only if **A** is consistent or maximally inconsistent, respectively.

Classical bounds for the Perron root of a nonnegative matrix are the minimum and maximum row sums (see, for example Theorem 8.1.22 in [9]):