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Probabilistic inferences from conjoined to iterated conditionals [☆,☆☆]

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ABSTRACT

There is wide support in logic, philosophy, and psychology for the hypothesis that the probability of the indicative conditional of natural language, $P(\textit{if } A \textit{ then } B)$, is the conditional probability of B given A, $P(B|A)$. We identify a conditional which is such that $P(\textit{if } A \textit{ then } B) = P(B|A)$ with de Finetti's conditional event, $B|A$. An objection to making this identification in the past was that it appeared unclear how to form compounds and iterations of conditional events. In this paper, we illustrate how to overcome this objection with a probabilistic analysis, based on coherence, of these compounds and iterations. We interpret the compounds and iterations as conditional random quantities which, given some logical dependencies, may reduce to conditional events. We show how the inference to $B|A$ from A and B can be extended to compounds and iterations of both conditional events and biconditional events. Moreover, we determine the respective uncertainty propagation rules. Finally, we make some comments on extending our analysis to counterfactuals.

1. Introduction

There is wide agreement in logic and philosophy that the indicative conditional of natural language, *if A then B*, cannot be adequately represented as the material conditional of binary logic, logically equivalent to $\overline{A} \vee B$ (*not-A or B*) [26]. Psychological studies have also shown that ordinary people do not judge the probability of *if A then B*, $P(\textit{if } A \textit{ then } B)$, to be the probability of the material conditional, $P(\overline{A} \vee B)$, but rather tend to assess it as the conditional probability of B given A, $P(B|A)$, or at least to converge on this assessment [5,28,30,55,70,71,83]. These psychological results have been taken to imply [5,29,41,64,71,72], that *if A then B* is best represented, either as the probability conditional of Adams [3], or as the *conditional event* $B|A$ of de Finetti [21,22], the probability of which is $P(B|A)$. We will adopt the latter view in the

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present paper and base our analysis on conditional events and coherence (for related analyses, specifically on categorical syllogisms, squares of opposition under coherence and on generalized argument forms see [42,77,76,82]). One possible objection to holding that $P(\textit{if } A \textit{ then } B) = P(B|A)$ is that it is supposedly unclear how this relation extends to compounds of conditionals and makes sense of them [24,26,87]. Yet consider:

$$\overbrace{\textit{She will be angry}}^{a} \text{ if } \overbrace{\textit{her son gets a } \mathrm{B}}^{b} \text{ and } \overbrace{\textit{she will be furious}}^{f} \text{ if } \overbrace{\textit{he gets a } \mathrm{C}}^{c}. \quad (1)$$

The above conjunction appears to make sense, as does the following seemingly even more complex conditional construction [24]:

If *she will be angry* if *her son gets a* B, then *she will be furious* if *he gets a* C. (2)

We will show below, in reply to the objection, how to give sense to (1) and (2) in terms of *compound conditionals*. Specifically, we will interpret (1) as a *conjunction* of two conditionals ($a|b$ and $f|c$) and (2) in terms of a *conditional* whose antecedent ($a|b$) and consequent ($f|c$) are both conditionals (if $a|b$, then $f|c$). But we note first that the *iterated conditional* (2) validly follows from the conjunction (1) by the form of inference we will call *centering* which, as we will show, can be extended to the compounds of conditionals (see Section 3 below). We point out that our framework is quantitative rather than a logical one. Indeed in our approach, syntactically conjoined and iterated conditionals in natural language are analyzed as conditional random quantities, which can sometimes reduce to conditional events, given logical dependencies ([47,50]). For instance, the biconditional event $A||B$, which we will define by $(B|A) \wedge (A|B)$, reduces to the conditional $(A \wedge B)|(A \vee B)$. Moreover, the notion of *biconditional centering* will be given.

The outline of the paper is as follows. In Section 2 we give some preliminaries on the notions of coherence and p-entailment for conditional random quantities, which assume values in $[0,1]$. In Section 3, after recalling the notions of conjoined conditional and iterated conditional, we study the p-validity of centering in the case where the basic events are replaced by conditionals. In Section 4 we give some results on coherence, by determining the lower and upper bounds for the conclusion of two-premise centering; we also examine the classical case by obtaining the same lower and upper bounds. In Section 5, after recalling the classical biconditional introduction rule, we present an analogue in terms of conditional events (*biconditional AND rule*); we also obtain one-premise and two-premise biconditional centering. In Section 6 we determine the lower and upper bounds for the conclusion of two-premise biconditional centering. In Section 7 we investigate reversed inferences (i.e., inferences from the conclusion to its premises), by determining the lower and upper bounds for the premises of the biconditional AND rule. Section 8 sketches how to apply results of this paper to study selected counterfactuals, and remark that the Import-Export Principle is not valid in our approach which allows us to avoid Lewis' notorious triviality results. Section 9 concludes with some remarks on future work. Further details which expand Section 2 are given in Appendix A.

2. Some preliminaries

The coherence-based approach to probability and to other uncertain measures has been adopted by many authors (see, e.g., [7,8,12–18,38,49,71,88]); we recall below some basic aspects on the notions of coherence and of p-entailment. In Appendix A we will give further details on coherence of probability and prevision assessments.

2.1. Events and constituents

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which is either true (T), or false (F). The indicator of an event A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false, respectively, and we use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\emptyset$ the impossible one (notice that, when necessary, the symbol $\emptyset$ will denote the empty set). Given two events A and B, we denote by $A \wedge B$ the logical intersection, or conjunction, of A and B; moreover, we denote by $A \vee B$ the logical union, or disjunction, of A and B. To simplify notations, in many cases we denote the conjunction of A and B (and its indicator) as AB; of course, AB coincides with the product of A and B. We denote by $\overline{A}$ the negation of A. Of course, the truth values for conjunctions, disjunctions and negations are obtained by applying the propositional logic. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\overline{B} = \emptyset$, which means that A and $\overline{B}$ cannot both be true.

Given n events $A_1, \dots, A_n$, as $A_i \vee \overline{A}_i = \Omega$, $i = 1, \dots, n$, by expanding the expression $\bigwedge_{i=1}^{n}(A_i \vee \overline{A}_i)$, we obtain

$$\Omega = \bigwedge_{i=1}^{n}(A_i \vee \overline{A}_i) = (A_1 \cdots A_n) \vee (A_1 \cdots A_{n-1}\,\overline{A}_n) \vee \cdots \vee (\overline{A}_1 \cdots \overline{A}_n);$$

that is the sure event Ω is represented as the disjunction of 2^n logical conjunctions. By discarding the conjunctions which are impossible (if any), the remaining ones are the *constituents* generated by $A_1, \dots, A_n$. Of course, the constituents are pairwise logically incompatible; then, they are a partition of Ω. We recall that $A_1, \dots, A_n$ are logically independent when