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Ambiguous games without a state space and full rationality ☆

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ABSTRACT

This work aims to differentiate and to better understand the assumptions that must be imposed on the structure of ambiguity and on the attitudes towards ambiguity in order to have existence of equilibria in games under ambiguous belief correspondences. In the present paper, this class of games is studied under substantially weaker assumptions on agents' preferences, as they are not required to be rational and therefore do not have any functional representation.

A new approach is required to deal with preferences that are not rational, in this particular framework; in fact, the present work shows that the attitudes of agents towards the imprecision of probabilistic beliefs play a key role in the issue of equilibrium existence, whenever they are combined with some property of convexity/concavity of the ambiguous belief correspondences.

The paper also studies the role played by these assumptions in different specific models (such as incomplete information games with multiple priors or games under strategic ambiguity), so as to illustrate the applicability of the results of equilibrium existence and connections with previous literature.

1. Introduction

As shown in the seminal paper by Ellsberg [14], in the problem of decision under uncertainty, beliefs cannot always be represented by conventional probabilities. There is substantial evidence, both from theory and applications, that *ambiguity* emerges even more clearly in game theory as the concept of equilibrium has a specific source of uncertainty in the expectations of players about their opponents behavior (strategic ambiguity). Therefore, it seems to be noteworthy to study games and their equilibrium concepts in case of ambiguity. This is the issue that this paper addresses; more precisely, this work looks at games in which uncertainty is described by sets of probability distributions and studies sufficient conditions for the existence of equilibria in this framework. The paper provides a generalization of different results already presented in the literature. In particular, the assumption of *completeness* and *transitivity* (usually called *rationality*) of preferences, that underlies the previous literature on ambiguous games,[1] is here removed. The present work points out that such issue entails

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[1] Indeed, even the literature on Decisions under Uncertainty provides few models of preferences that are not "fully rational" in this sense; *Interval Dominance* is perhaps the most known example.

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a new approach, as it is shown below that the attitudes of players towards the imprecision of probabilistic beliefs play a key role to obtain the minimal convexity assumption for preferences that is required in the equilibrium existence theorems (whenever some additional assumption on the representation of ambiguous beliefs are imposed).

In order to clarify where the contribution of this work lies, it seems appropriate to relate the approach of this paper to two important issues which emerge more or less explicitly in the literature on decision theory under ambiguity. A first issue is the "structure of ambiguity", that is, the way ambiguous beliefs are represented, regardless of the preference of the agent. The *theory of imprecise probabilities* (see, for instance, [34] for a recent survey) studies in which way ambiguity can be represented by probability judgments and it analyzes the relations between the different representations. The other issue deals with attitudes of the agents towards ambiguity; optimism and pessimism, in the broad sense, are recognized as the most significant behavioral traits related to ambiguity; essentially, they depend on whether or not the agent expects that ambiguity will be resolved in his favor. In [17], Gilboa and Marinacci survey some of the main findings in the decision theory literature devoted to the axiomatization of different preference relations over the set of ambiguous alternatives. In this strand of literature, the source of uncertainty is described by an underlying state space and decision makers are endowed with preferences over a set of alternatives, called *acts*, i.e. functions from the state space to a space of *consequences*. Representation theorems for preferences axiomatize the fact that the decision maker is endowed with a utility function which gives a numerical outcome for every possible consequence, and with a probability judgment on the state space.[2] In this particular framework, the probabilistic belief is completely endogenous and depends only on the attitude of the decision maker towards ambiguity. However, there is another strand of research that is devoted to the axiomatization of preferences in case ambiguous alternatives have an exogenous or objective probabilistic representation (see, for instance, the survey by [33] or [32] and [7] or [21] for recent developments and references therein). In [15], the two approaches are merged: in a classical decision making problem with a state space, they look at preferences over pairs (P, f) where P is a set of probability distributions over the state space (representing the objective information available) and f is an act. They provide a maxmin representation for complete preferences in this setting which has an important feature: a pessimistic attitude towards the imprecision of the probability judgments (i.e. imprecision aversion) is behind the Gilboa and Schmeidler's definition of *ambiguity aversion* and the maxmin representation.

Aim of this paper is to differentiate and to better understand the assumptions that must be imposed on the structure of ambiguity and on the attitudes towards ambiguity in order to have equilibrium existence in a game under ambiguity. In particular, it will be shown that a notion of imprecision aversion and its optimistic counterpart are key assumptions for the existence of equilibria. The approach considered in this work is different from the papers which investigate the effects of uncertainty aversion in incomplete information games with multiple priors.[3] In the present paper, ambiguous games are, instead, regarded from a different perspective as the approach here follows another strand of research in decision theory (proposed in [1], [25] and [31]), in which the decision-maker, facing ambiguity, is not able to understand what the relevant states are, so that the information available can be expressed entirely in the space of lotteries (probabilities) over consequences. This approach seems particularly useful to study ambiguity in games. In fact, on the one hand, the model *without the state space* and the classical *multiple prior* model can be reconciled as an ambiguous act can be evaluated by its induced set of distributions over consequences (see [1] and [25]). On the other hand, game theory provides further evidence that ambiguity cannot always be reconducted to the classical approach with a state space and multiple priors. The literature on ambiguous games (see for instance [12], [23], [20], [13], [24], [22], [26] and [6]) has shown that the classical equilibrium notion embodies a specific source of ambiguity: In equilibrium, players choose their optimal strategies provided that they have correct expectations about the behavior of their opponents. However, agents may have ambiguous beliefs about opponents' strategy choices; for instance, such ambiguous beliefs arise from the fact that there are multiple Nash equilibria in a large class of games. Therefore, rational agents take this issue into account and face a problem of decision making under ambiguity.[4]

Previous papers, [8], [9], [10], introduced and studied the (so called) model of *game under ambiguous belief correspondences*[5] which provides a rather general tool to study ambiguity in games. The key point of these papers is that, for every player, ambiguity is directly represented by a belief correspondence which maps the set of strategy profiles into the set of all subsets of probability distributions over the outcomes of the game. For each player and for every given strategy profile, the belief correspondence gives the set of probability distributions over the possible outcomes of the game that the corresponding player perceives to be feasible and consistent with the actual strategy profile. It follows that belief correspondences might represent exogenous ambiguity as done in [1] and [25]; but, at the same time, it turns out (see the examples in [9] and also Section 6 below) that many existing models of ambiguous game have an equivalent formulation in terms of

[2] In the *multiple prior* approach, firstly studied in [18], the probability judgment is provided by a convex set of probability distributions. In the Choquet Expected Utility approach, firstly introduced by [30], the probability judgment is given by a convex capacity.

[3] [19] first consider this approach, [5] considers games à la Aumann under more general preferences. [4] characterize equilibrium existence in terms of the preferences of the players; the evidence from this paper is that equilibria exist if and only if agents are ambiguity averse, as ambiguity aversion is deeply related to some form of convexity of preferences.

[4] There is no evidence in the literature showing that this kind of ambiguity can be properly reconducted to incomplete information games à la Harsanyi (that is, with a state space) under multiple priors which, in turn, must be generalized in order to encompass this specific game theoretical issue.

[5] [8] presents the general model, an existence theorem and many motivating examples. Stability of the equilibria is studied in [9]. Finally, [10] extends the model to the case of variational preferences.